DIAMETER AND CONNECTIVITY CONSIDERATIONS IN THE DESIGN OF COMMUNICATION NETWORKS

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

A communication network is modeled by an undirected graph without loops and multiple edges. The maximal message delay index, in the network, is expressed in terms of the diameter of the graph. Three reliability measures are considered: a given minimal degree, edge-connectivity and vertex connectivity.

Let $H_1(k,d)$ be the class of all graphs with diameter d and minimal degree k, $H_2(k,d)$ is a subclass of $H_2(k,d)$ consisting of all k-edge connected graphs in $H_1(k,d)$ is the subclass of all k-vertex connected graphs in $H_2(k,d)$. $H_i(n,k,d)$

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20. consists of the graphs in $H_i(k,d)$ with exactly n vertices (i=1,2,3). Let $f_i(k,d)$, $g_i(k,d)$ be the minimum number of vertices and edges, respectively, that an $H_i(k,d)$-graph must have and let $g_i(n,k,d)$ be the minimal number of edges of an $H_i(n,k,d)$-graph (i=1,2,3). In Chapter 2 and 3 our main concern is to calculate the values of $f_i(k,d)$, $g_i(k,d)$ and $g_i(n,k,d)$ for arbitrary natural numbers n,d,d (i=1,2,3). Furthermore, graphs attaining the minimal number of vertices and edges are constructed.

Motivated by the problem of designing communication networks whose maximal message delay does not exceed a prescribed value, even if a number of communication links fail, we define a new class of graphs. A graph G is called an (ℓ,d)-graph if the removal of at least ℓ edges from G is required in order that the resulting graph would have a diameter larger than d. G is called ℓ-distance stable if the removal of at least ℓ edges from G is required to increase the distance between any pair of nonadjacent vertices of G. In Chapter 4, classes of (ℓ,d)-graphs and ℓ-distance stable graphs are constructed and various properties of these graphs are given. In particular, we obtain necessary and sufficient conditions for graphs to belong to some special classes of $(2,d)$-graphs and a Menger type theorem for ℓ-distance stable graphs. Finally, we consider some extremal problems related to $(2,d)$-graphs of diameter d, called 2-diameter stable graphs. More specifically, the minimal number of vertices and edges of a 2-diameter stable graph of diameter d is obtained and bounds on the minimal number of edges of a 2-diameter stable graph on n vertices are calculated.

DIAMETER AND CONNECTIVITY CONSIDERATIONS

IN THE DESIGN OF COMMUNICATION NETWORKS

by

Jehuda Hartman

Principal Investigator: Izhak Rubin

Department of System Science
School of Engineering and Applied Science
University of California
Los Angeles, California

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ABSTRACT

A communication network is modeled by an undirected graph without loops and multiple edges. The maximal message delay index, in the network, is expressed in terms of the diameter of the graph. Three reliability measures are considered: a given minimal degree, edge-connectivity and vertex connectivity.

Let $H_1(k,d)$ be the class of all graphs with diameter d and minimal degree k, $H_2(k,d)$ is a subclass of $H_2(k,d)$ consisting of all k-edge connected graphs in $H_1(k,d)$ and $H_3(k,d)$ is the subclass of all k-vertex connected graphs in $H_2(k,d)$. $H_i(n,k,d)$ consists of the graphs in $H_i(k,d)$ with exactly n vertices (i=1,2,3). Let $f_i(k,d)$, $g_i(k,d)$ be the minimum number of vertices and edges, respectively, that an $H_i(k,d)$-graph must have and let $g_i(n,k,d)$ be the minimal number of edges of an $H_i(n,k,d)$-graph (i=1,2,3). In Chapter 2 and 3 our main concern is to calculate the values of $f_i(k,d)$, $g_i(k,d)$ and $g_i(n,k,d)$ for arbitrary natural numbers n,k,d (i=1,2,3). Furthermore, graphs attaining the minimal number of vertices and edges are constructed.

Motivated by the problem of designing communication networks whose maximal message delay does not exceed a prescribed value, even if a number of communication links fail, we define a new class of graphs. A graph G is called an (ℓ,d)-graph if the removal of at least ℓ edges from G is required in order that the resulting graph would have a diameter larger than d. G is called ℓ-distance stable if the removal of at least ℓ edges from G is required to increase the distance between any pair of nonadjacent vertices of G. In Chapter 4, classes of

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(ℓ,d)-graphs and ℓ-distance stable graphs are constructed and various properties of these graphs are given. In particular, we obtain necessary and sufficient conditions for graphs to belong to some special classes of $(2,d)$-graphs and a Menger type theorem for ℓ-distance stable graphs. Finally, we consider some extremal problems related to $(2,d)$-graphs of diameter d, called 2-diameter stable graphs. More specifically, the minimal number of vertices and edges of a 2-diameter stable graph of diameter d is obtained and bounds on the minimal number of edges of a 2-diameter stable graph on n vertices are calculated.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

In this dissertation, a communication network is topologically
modeled by a finite linear graph, whose vertices represent the stations
(transmitting, receiving or relying information) while the edges repre-
sent the communication channels. Computer-communication, satellite and
telephone networks are examples.

In designing a communication network, reliability (survivability,
invulnerability) and message delays are of prime importance as perfor-
mance indices. In studies of communication networks, reliability has
been defined in various ways (See [28]). The most common definition
assumes the network to be operational, under channel or station failures,
provided there exists a communication path between any pair of stations.
Thus, under the latter definition even if a set of stations whose
number does not exceed a prescribed value fail, it is still possible
to form communication paths between all pairs of functioning stations.
The latter measure of network reliability is called in graph theoretical
terms the vertex connectivity of the underlying graph. Similarly, one
can use the edge connectivity of the underlying graph as a reliability
measure and require that even if a set of communication links, whose
number does not exceed a prescribed value, fail, it is possible to
communicate between any pair of stations in the network. A weaker form
of reliability is the requirement that each station should be directly
linked to at least a given number of stations. All these three reli-
ability measure are considered in this research.

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Messages, arriving at random at the network source terminals and routed through the network towards a destination terminals, experience queueing and transmission time delays. The maximal delay experienced by a message flowing through the network can be measured in terms of the diameter of the underlying graph (representing the maximal distance between any two vertices of the graph). The latter clearly yields the maximal message delay if the message delays across the channels in the network are of comparable value. Furthermore, for any store-and-forward communication network, under a fixed routing discipline, it has been shown [1] that the product of the prescribed maximal message delay γ and the associated minimal overall network capacity C, is characterized by a unique Delay-Capacity product function (γC). The latter is shown in [1] to be the sum of two terms. The first term is $\gamma\lambda_I$, where λ_I is the overall internal traffic flow. The second term, called the Delay-Capacity Product number, $(\gamma C)^*$, is uniquely determined by the routing discipline and the topological structure of the communication network, and is independent of the terminal traffic intensities. It is readily observed that for any network with m lines and diameter d, by assigning equal delays across the channels, one obtains $(\gamma C)^* \leq$ md. Furthermore, it has been shown [3], [4] that $(\gamma C)^* =$ md for many networks under a variety of general routing disciplines, while for other cases md serves as a tight bound. One also notes that the overall internal flow λ_I can be expressed as $\lambda_I = \bar{n}\lambda_E$, where λ_E is the prescribed multiterminal flow value (network throughput) while $\bar{n}$ denotes the average route length. Generally, $\bar{n} \leq$ d. Furthermore, for a uniform traffic matrix (and other situations) $\bar{n}$ is proportional to m. Hence, if the diameter

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d of a network is prescribed, the topological structure which yields the minimal (γC) product needs to have the minimal number of lines. Consequently, we study the characteristics of reliable (as expressed by a prescribed number of minimal links joined to each vertex, vertex connectivity or edge connectivity) graphs with minimal number of lines having a prescribed diameter. The latter will represent reliable topological structures for communication networks, attaining the minimal delay-capacity product, under a diameter (delay, maximal number of message hops) constraint. For perturbation techniques used for the design of computer communication networks with connectivity and diameter constraints, see Lavia and Manning [25].

The use of connectivity indices of the underlying graph as a reliability measure is based on the assumption the network is operational in the presence of failures provided there is at least one communication path remaining between any pair of stations in the network. However, under failures the resulting network may have an excessively large diameter, which may result in intolerable queueing delays while routing a message through the network. We thus study here graphs whose diameter does not exceed a prescribed value even if a number of communication links fail. In graph theoretical terms, the underlying graph of such a network will defined to be diameter-stable. Properties of diameter-stable graphs, and diameter stable graphs having the minimal number of vertices, are investigated in the sequel.

Extremal graphs of diameter two with prescribed minimum degree were studied by Bondy and Murty [14]. Studies [10] and [11] deal with connectivity problems without the diameter constraint. Some properties

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of graphs with prescribed connectivity and diameter are studied in [7], [8], [12], [13], [14], [25] and [27]. Special problems associated with extremal diameter stable graph (having mainly diameters 2, 3 and 4) were considered by Bollobás, Murty, Vijayan and Caccetta in [15]-[22], while in [23] vertex distance stability problems are studied. For an extensive summary of methods of analysis and design of communication network, the reader is refered to [28]. Further references to the above-mentioned papers are made in the appropriate sections of this work.

1.2 Terminology and Notation

All graphs considered in this paper are undirected, without loops and multiple edges. By $V(G)$ and $E(G)$ we denote the set of vertices and the set of edges of the graph G, respectively. The number of elements of a set A is denoted by $|A|$. The <u>degree</u> of a vertex $v \in V(G)$ is defined as the number of vertices adjacent to v, and is denoted by $deg(v)$. A graph all whose vertices have the same degree k is called a <u>k-regular</u> graph. An <u>almost k-regular</u> graph is a graph which has one vertex of degree $k + 1$, while all the other vertices are of degree k. The edge with end-vertices v and w is denoted by vw.

A graph G with $|V(G)| \geq k + 1$ is called <u>k-vertex connected</u>, or simply <u>k-connected</u> (<u>k-edge connected</u>) if between any pair x, y of distinct vertices of G, there are at least k vertex (edge) disjoint x, y-paths in G. It is obvious that a k-connected (k-edge connected) graph cannot be disconnected by removing less than k vertices (edges) from the graph. The converse is also true. Hence, a graph is k-connected (k-edge connected) if and only if $|V(G)| \geq k + 1$ and it is

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impossible to disconnect G by removing $k - 1$ or fewer vertices (edges) from G, (Menger-Whitney theorem and the corresponding edge version by Ford, Fulterson and others, see [9] Chapter 5).

The _distance_ $d_G(x,y)$ between two vertices $x, y \in V(G)$ is the length of the shortest path in G joining x and y. In cases where no confusion can occur we may omit the index G from the function $d_G(x,y)$. The diameter of G, $d(G)$, is defined as

$$d(G) \triangleq \max_{x,y \in V(G)} d_G(x,y) .$$

A pair of vertices $x,y \in V(G)$ such that $d_G(x,y) = d(G)$ is called a _diametrical pair_ of vertices.

In many cases in the following chapters we will have a function $g(n,k,d)$ (where n denotes the number of vertices of a graph G, d is the diameter and k is some reliability measure) bounded by an upper and a lower bound, which will also be functions of n, k and d. To estimate the tightness of the inequality a _tightness measure_ will be applied to the inequality by dividing each bound by n and taking the limit as $n \to \infty$. The latter yield per vertex asymptotic measures.

K_n, $K_{m,n}$ and C_n will denote the complete graph on n vertices, the complete bipartite graph on n and m vertices and the cycle on n vertices, respectively. [x] denotes, as usual, the integral value of a real number x. For further definitions used in this dissertation the reader is refered to [9].

1.3 Outline

Most of the work reported in this dissertation is concerned with synthesis of graphs under reliability and diameter constraints. While

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design results were at time the main objective, analysis of certain classes of graphs are also presented.

Chapter 2 is concerned with constructions of graphs with prescribed diameter d, and minimum degree k, called $H_1(k,d)$-graphs. The minimal number of vertices and edges that an $H_1(k,d)$-graph must have, is calculated and classes of extremal graphs, in this sense, are synthesized. Then, we consider $H_1(n,k,d)$-graphs, which are $H_1(k,d)$-graphs with exactly n vertices, and obtain bounds on the minimal number of edges that an $H_1(n,k,d)$-graph must have.

In Chapter 3 graphs with prescribed diameter d and connectivity (edge-connectivity) k, called $H_3(k,d)$-graphs ($H_2(k,d)$-graphs) are considered. The minimal numbers of vertices and edges of an $H_3(k,d)$-graph are obtained, extremal graphs are constructed and the family of $H_3(k,d)$-graphs having n vertices is studied. In section 3.4, some results concerning $H_2(k,d)$-graph are given.

In Chapter 4 we study diameter stable graphs. A graph G is called an (ℓ,d)-graph if the removal of at least ℓ edges from G is required for the resulting graph to have a diameter larger than d. $(\ell,d(G))$-graphs are called ℓ-diameter stable graphs. A graph G with the property that at least ℓ edges are to be removed from G in order to increase the distance between any pair of non-adjacent vertices of G, is called an ℓ-distance stable graph. Classes of (ℓ,d)-graphs, ℓ-diameter stable graphs and ℓ-distance stable graphs are constructed for any arbitrary ℓ, $d \geq 2$, in Section 4.2. Various properties of the latter classes of graphs are presented in Section 4.3. In particular, a Mengerian type characterization of ℓ-distance stable graphs,

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indicating the appropriate similarity of ℓ-distance stable graphs to ℓ-edge connected graphs, is obtained. In Section 4.4, (2,d)-graphs are considered. Two classes of 2-diameter stable graphs are characterized by necessary and sufficient conditions. In Section 4.5, a few extremal problems, similar to those in Chapters 1 and 2, are solved for 2-diameter stable graphs.

Finally, in Chapter 5, we conclude with summary of the work presented, indicating further problems for future research.

CHAPTER 2

GRAPHS WITH PRESCRIBED DIAMETER

AND MINIMUM DEGREE

2.1 Introduction

Consider a communication network of stations in which certain pairs of vertices are linked directly and other pairs must communicate indirectly by means of a sequence of direct links. The given communication network is represented as usual by a graph G whose vertices and edges correspond respectively to the stations and the direct links of the communication network. Assume that if a failure occurs at a station it can rely for support only upon those stations to which it is directly linked. Reliability considerations may require therefore that each station should be directly linked to at least k stations. Furthermore, in order to have a reasonable message delay when it is routed over the network, we may require that each pair of stations must be able to communicate by means of a sequence of direct links which does not exceed a given integer $d \geq 1$. In graph theoretical terminology, it is required to construct a graph with prescribed minimal degree of the vertices — k, and a given diameter — d.

Let $H_1(k,d)$ denote the class of all graphs with minimal degree k and diameter d. The subclass of graphs in $H_1(k,d)$ having exactly n vertices is denoted by $H_1(n,k,d)$. The graphs in $H_1(k,d)$ $(H_1(n,k,d))$ are called $\underline{H_1(k,d)\text{-graphs}}$ $(\underline{H_1(n,k,d)\text{-graphs}})$.

Let

$$f_1(k,d) \overset{\Delta}{=} \underset{G \in H_1(k,d)}{\text{Min}} |V(G)|,$$

$$g_1(k,d) \overset{\Delta}{=} \underset{G \in H_1(k,d)}{\text{Min}} |E(G)|,$$
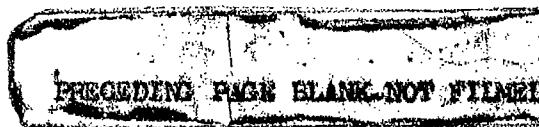
$$g_1(n,k,d) \overset{\Delta}{=} \underset{G \in H_1(n,k,d)}{\text{Min}} |E(G)|.$$

$H_1(k,d)$-graphs with $f_1(k,d)$ vertices are called <u>vertex extremal graphs</u> of $H_1(k,d)$, and $H_1(k,d)$-graphs ($H_1(n,k,d)$-graphs), having $g_1(k,d)$ ($g_1(n,k,d)$) edges are simply called <u>extremal graphs</u> of $H_1(k,d)$ ($H_1(n,k,d)$).

In this chapter we investigate the above functions and find some extremal graphs.

Considering the case $d = 2$, J. A. Bundy and U. S. R. Murty [14] proved that if $n > k^3 + \alpha(n) \cdot \alpha(k) \cdot k + 1$ (where for an integer t, $\alpha(t) = 0$ or 1 according as t is odd or even), then

$$g_1(n,k,2) = \left[\frac{(n-1)(k+1) + 1}{2} \right]$$

and every $H_1(n,k,2)$-extremal graph has a vertex of degree $n - 1$. They have also obtained a characterization of vertex extremal $H_1(n,k,2)$-graphs.

2.2 Vertex Extremal $H_1(k,d)$-graphs

Assume G to be a graph of diameter d, then there exists a diametrical path x_0, x_1, ..., x_d in G, ($x_i \in V(G)$).

Define w.r. to the vertex x_0, $V_i \overset{\Delta}{=} \left\{ v \in V(G) \mid d(v,x_0) = i \right\}$, $0 \le i \le d$. Clearly $V_0 = \{x_0\}$, $x_1 \in V_i$ $1 \le i \le d$.

Let

$$n_i \stackrel{\Delta}{=} |V_i| \ , \quad n \stackrel{\Delta}{=} |V(G)| = \sum_{i=0}^{d} n_i.$$

If in addition the minimal degree of G is k, we must have;

$$n_{i-1} + n_i + n_{i+1} \geq k + 1, \quad 2 \leq i \leq d - 2. \tag{2.1}$$

$$n_0 + n_1 \geq k + 1 \ , \quad n_{d-1} + n_d \geq k + 1. \tag{2.2}$$

$$n_0 + n_1 + n_2 \geq k + 2 \ , \quad n_{d-2} + n_{d-1} + n_d \geq k + 2. \tag{2.3}$$

Inequality (2.1) follows by noting that $\deg (v) \geq k, \forall v \in V(G)$, while (2.2) and (2.3) follow from $\deg (x_o) \geq k$, $\deg (x_d) \geq k$. The following theorem determines $f_1(k,d)$.

Theorem 2.1. For all integers $k \geq 2$, $f_1(k,1) = k+1$, $f_2(k,2) = k + 2$ and for $\ell \geq 1$,

$$f_1(k, 3\ell + i) = (k+1)(\ell+1) + i \quad i = 0, 1, 2. \tag{2.4}$$

Proof. It is easy to verify $f_1(k,1) = k + 1$ and $f_2(k,2) = k + 2$. For $d > 2$ we examine three cases of $d \pmod 3$.

Case 1: $d = 3\ell \ , \quad \ell \geq 1$.

By (2.2) $n_o + n_1 + n_{d-1} + n_d \geq 2(k+1)$, and if $\ell > 1$ then by (2.1)

$$\sum_{\substack{j=3m \\ 1 \leq m \leq \ell-1}} (n_{j-1} + n_j + n_{j+1}) \geq (k + 1) (\ell - 1).$$

Hence, $\quad n = \displaystyle\sum_{i=0}^{d} n_i \geq 2(k+1) + (k+1)(\ell-1)$

and,

$$f_1(k, 3\ell) \geq (k+1)(\ell+1). \tag{2.5}$$

We now construct an $H_1(k, 3\ell)$-graph with $(k+1)(\ell+1)$ vertices,

Choose $\quad n_0 = 1, \; n_1 = k, \; n_2 = 1, \; n_3 = k-1, \; n_4 = 1, \; \ldots$

$$n_{3j+2} = 1, \; n_{3j+3} = k-1, \; n_{3j+4} = 1, \; \ldots$$

$$n_{d-1} = k, \; n_d = 1.$$

Two distinct vertices $p \in V_i$, $q \in V_j$ are joined by an edge if and only if $|\; i-j \;| \le 1$, (Figure 2.1). The graph obtained is obviously an $H_1(k, 3\ell)$-graph.
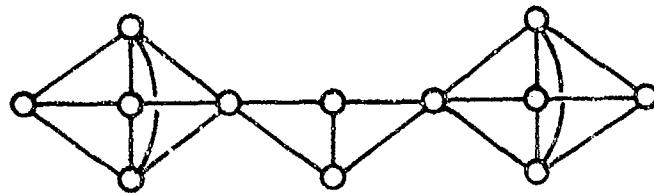
Figure 2.1. A Vertex Extremal $H_1(3,6)$ – Graph.

Thus, by (2.5)

$$f_1(k, \; 3\ell) = (k+1)(\ell+1). \tag{2.6}$$

$\underline{\text{Case 2}}:$ $\quad d = 3\ell + 1$, $\quad \ell \ge 1$. By (2.2) and (2.3),

$$n_0 + n_1 + n_{d-2} + n_{d-1} + n_d \ge 2(k+1) + 1,$$

and if $\ell > 1$ by (2.1)

$$\sum_{\substack{j=3m \\ 1 \le m \le \ell-1}} (n_{j-1} + n_j + n_{j+1}) \ge (k+1)(\ell-1)$$

Hence,

$$n = \sum_{i=0}^{n} n_i \ge 2(k+1) + 1 + (k+1)(\ell-1)$$

and,

$$f_1(k, \; 3\ell + 1) \ge (k+1)(\ell+1) + 1. \tag{2.7}$$

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An $H_1(k, 3\ell + 1)$-graph with $(k+1)(\ell+1) + 1$ vertices may be constructed as follows.

Choose,

$$n_o = 1, \; n_1 = k, \; n_2 = 1, \; n_3 = k-1, \; n_4 = 1, \; \ldots$$

$$n_{3j+2} = 1, \; n_{3j+3} = k-1, \; n_{3j+4} = 1, \; \ldots$$

$$n_{d-2} = 1, \; n_{d-1} = k, \; n_d = 1.$$

As in Case 1, join $p \in V_i$, $q \in V_j$ by an edge if and only if $|\, i-j \,| \leq 1$, (Figure 2.2).
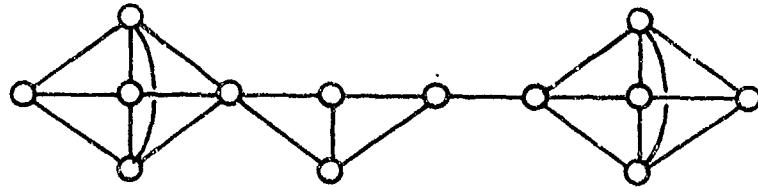
Figure 2.2. A Vertex Extremal $H_1(3,7)$ – Graph.

This construction shows equality in (2.7) is attained, thus,

$$f_1(k, \; 3\ell + 1) = (k+1)(\ell+1) + 1. \tag{2.8}$$

Case 3: $d = 3\ell + 2$, $\ell \geq 1$.

By (2.3)

$$n_o + n_1 + n_2 + n_{d-2} + n_{d-1} + n_d \geq 2(k+2),$$

and if $\ell > 1$ by (2.1)

$$\sum_{\substack{j=3m \\ 1 \leq m \leq \ell-1}} (n_{j-1} + n_j + n_{j+1}) \geq (k+1)(\ell-1).$$

As before,

$$f_1(k, \; 3\ell + 2) \geq (k+1)(\ell+1) + 2. \tag{2.9}$$

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To construct the vertex extremal graph in this case, take

$$n_o = 1, \ n_1 = k, \ n_2 = 1, \ \ldots$$

$$n_{3j} = 1, \ n_{3j+1} = k-1, \ n_{3j+2} = 1, \ \ldots$$

$$n_{d-2} = 1, \ n_{d-1} = k, \ n_d = 1,$$

and proceed as before, (Figure 2.3).
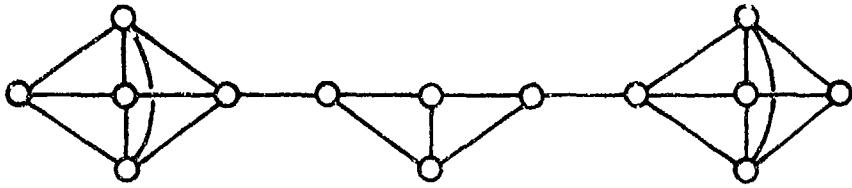
Figure 2.3. A Vertex Extremal H_1 (3,8) - Graph.

Hence from (2.9),

$$f_1(k, \ 3\ell+2) \ = \ (k+1)(\ell+1) + 2. \tag{2.10}$$

(2.4) follows from (2.6), (2.8) and (2.10).

Q.E.D.

2.3 Some Results for $g_1(k,d)$

Clearly,

$$g_1(k,d) \geq \frac{f_1(k,d) \cdot k}{2} . \tag{2.11}$$

We shall show that $g_1(k,d)$ differs from the right hand side of (2.11) by at most 1. As before we again treat three cases of $d \pmod 3$.

Theorem 2.2. For $k \geq 3$, $d \geq 2$, we have

$$\frac{f_1(k,d) \cdot k}{2} + 1 \geq g_1(k,d) \geq \left\lceil \frac{f_1(k,d) \cdot k + 1}{2} \right\rceil . \tag{2.12}$$

For $k \equiv 1 \pmod 2$ and $d \equiv 1, \ 2 \pmod 3$ the lower bound is attained, so that

$$g_1(k,d) = \left\lceil \frac{f_1(k,d) \cdot k + 1}{2} \right\rceil . \tag{2.13}$$

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<u>Proof</u>. In all the three cases of $d \pmod 3$ n_i's are defined and chosen as in the proof of Theorem 2.1, and two distinct vertices $p \in V_i$, $q \in V_j$ $(2 \leq i \leq j \leq d-2)$ are joined by an edge if and only if $|i-j| \leq 1$. Further, join by an edge all the vertices of V_1 and V_{d-1} to V_0 and V_d, respectively. Finally join an arbitrary vertex of V_1 to x_0 and an arbitrary vertex of V_{d-1} to x_d, so that the resulting graph is connected. Call the graph obtained $\bar{G}$, and distinguish between three cases of $d \pmod 3$. In each case we will complete $\bar{G}$, by addition of lines, to an $H_1(k,d)$-graph.

Case 1: $d = 3\ell$, $\ell \geq 1$.

The partial graph $\bar{G}$ constructed in the previous paragraph can be completed to a k-regular $H_1(k,d)$-graph (which, of course, will yield an $H_1(k,d)$-extremal graph), if and only if it is possible to construct on V_1 (and V_{d-1}) a graph G_1 with k-1 vertices of degree k-1 and a single vertex of degree k-2. But then the sum of degrees of the vertices of G_1 is $(k-1)^2 + (k-2)$, which is an odd integer $\forall k$, and therefore such a G_1 cannot exist. Instead, take on V_1 (and V_{d-1}) a complete graph on k vertices, K_k. The graph obtained is obviously an $H_1(k,d)$-graph, and it has $\frac{(k+1)(\ell+1) \cdot k}{2}$ edges, (Figure 2.4). Therefore,

$$\frac{(k+1)(\ell+1)k}{2} + 1 \qquad g_1(k,3\ell) \qquad \frac{(k+1)(\ell+1)k}{2} . \qquad (2.14)$$
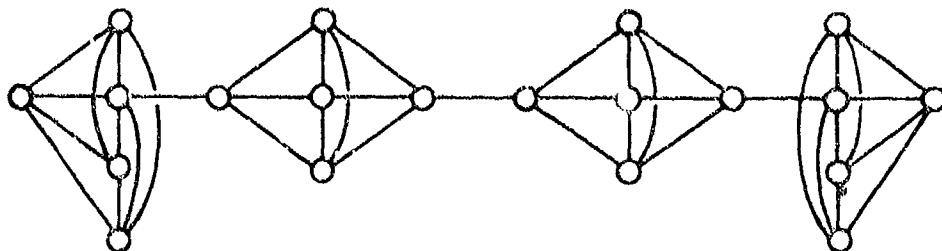
Figure 2.4. An $H_1(4,9)$ - Graph.

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(2.14) proves (2.12) for case 1.

Case 2: $d = 3\ell + 1, \ell \geq 1$.

Complete the partial graph $\overline{G}$ as follows. As in case 1 join by an edge any two distinct vertices of V_1, and join also k-1 arbitrary vertices of V_{d-1} to V_{d-2}. We would like to construct on V_{d-1} a graph G_{d-1} with k-1 vertices of degree k-2 and a single vertex of degree k-1. This is possible only if the degree sequence $(k-1, \underbrace{k-2, k-2, \ldots, k-2}_{k-1 \text{ times}})$ is graphical (which means that there exists a graph having the given degree sequence), (see Chapter 6 [9] and [26]). By Hakimi's Theorem [26], such a sequence is graphical if and only if the sum of the degrees $(k-1)^2$ is even, and the degree sequence, $\underbrace{(k-3, k-3, \ldots k-3)}_{k-1 \text{ times}}$ is graphical. $(k-1)^2$ is even if and only if k is odd, and a (k-3)-regular graph on k-1 vertices exists for k odd, (see [7], [8], [10]). Therefore, if $k = 2m+1$, $(m \geq 1)$, then the final graph obtained has $\frac{((2m+2)(\ell+1) + 1)(2m+1) + 1}{2}$ edges, and if G_{d-1} is constructed as in [7] and [8], the graph is clearly an $H_1(2m+1, 3\ell+1)$-graph, (Figure 2.5). By (2.4) and (2.11),

$$(m+1)((\ell+1)(2m+1) + 1) \geq g_1(2m+1, 3\ell+1) \geq \frac{((2m+2)(\ell+1) + 1)(2m+1)}{2} \qquad (2.15)$$

Since the lower bound in (2.15) is not a whole number, whereas the upper bound is a whole number differing from the lower bound by $\frac{1}{2}$, we conclude

$$g_1(2m+1, 3\ell+1) = (m+1)((\ell+1)(2m+1) + 1), \qquad (2.16)$$

which proves (2.13) for $k \equiv 1 \pmod 2$ and $d \equiv 1 \pmod 3$.
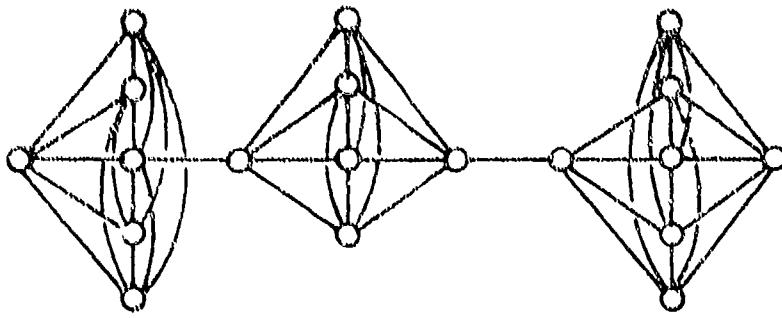
Figure 2.5. An $H_1(6,7)$ – Extremal Graph.

If on the other hand k is even ($k = 2m$, $m \geq 1$), construct on V_{d-1} a $(k-2)$-regular graph on k vertices as before, and addition of a single line will satisfy the degree requirements, (Figure 2.6). This settles (2.12) for $k = 0 \pmod 2$.
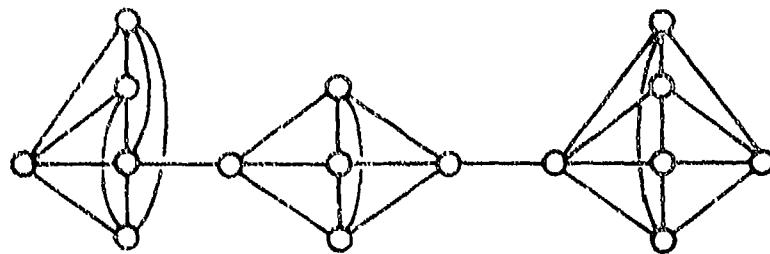
Figure 2.6. An $H_1(4,7)$ – Graph.

<u>Case 3</u>: $d = 3\ell + 2$, $\ell \geq 1$.

To complete the partical graph $\overline{G}$, join by an edge $k-1$ arbitrary vertices of V_1 and V_{d-1} to V_2 and V_{d-2} respectively.

If $k \equiv 1 \pmod 2$ we establish on V_1 and V_{d-1} a graph with one vertex of degree $k-1$ and $k-1$ vertices of degree $k-2$, (as in Case 2, this is

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possible if k is odd). Let $k = 2m + 1$ ($m \geq 1$) then we obtain $(2m+1)$-regular $H_1(2m+1, 3\ell+2)$-graph which is obviously an extremal graph, (Figure 2.7). Thus,

$$g_1(2m+1, \; 3\ell+2) = ((m+1)(\ell+1)+1)(2m+1), \qquad (2.17)$$

which proves (2.13) for $k \equiv 1 \pmod 2$ and $d \equiv 2 \pmod 3$.
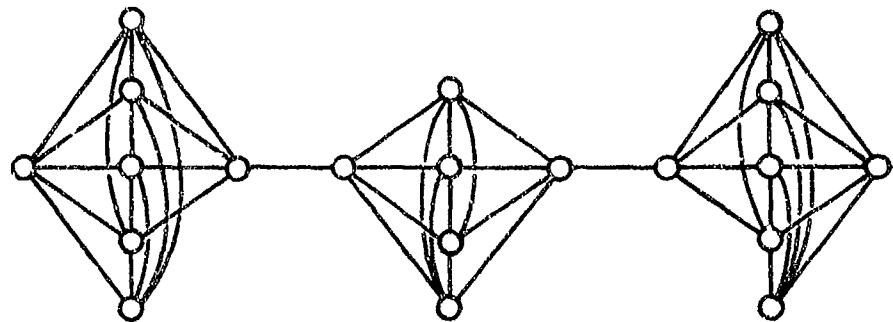
Figure 2.7. A 5-Regular $H_1(5,8)$ - Graph.

If on the other hand $k \equiv 0 \pmod 2$, take in V_1 and V_{d-1} a $(k-2)$-regular graph on k vertices as in Case 2, ($k \equiv 0 \pmod 2$), and add two additional edges to satisfy the degree requirements, (Figure 2.8).
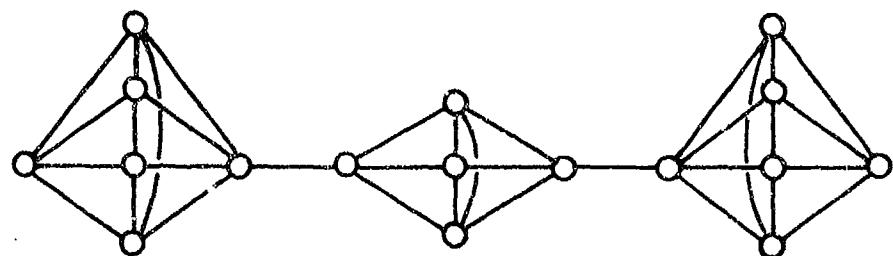
Figure 2.8. An $H_1(4,8)$ - Graph.

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This completes the proof of Theorem 2.4.

Q.E.D.

It should be noted that all the extremal-$H_1(k,d)$ graphs obtained in Section 2.3, are obviously also vertex extremal $H_1(k,d)$-graphs.

2.4 Inequalities Concerning the Class $H_1(n,k,d)$

First we note that if the diameter is not prescribed and a graph on n vertices whose minimal degree is k, is to be constructed, then a k-regular graph or an almost k-regular graph on n vertices can always be obtained.

Lemma 2.3. For all integers $n > k > 0$, there exists a k-regular, or almost k-regular graph on n vertices.

One way to construct such graphs is given in [10], where k-connected graphs on n vertices having minimal number of edges are obtained. The graphs of [10], have degrees $\geq k$ and satisfy our requirement here, but their diameter is not prescribed.

As mentioned before $g_1(n,k,2)$ was computed in [14]. We consider thus here $H_1(n,k,d)$ with $d \geq 3$.

Clearly,

$$g_1(n,k,d) \geq g_1(k,d)$$

for

$$n \geq f_1(k,d).$$

Considering a graph in $H_1(n,k,d)$, we derive in the next theorem relations between the parameters n,k and d.

Theorem 2.4. If there exist graphs in $H_1(n,k,d)$, $k,d \geq 1$,

with $\quad d = 3\ell + i \ (\ell \geq 1, \ 0 \leq i \leq 2), \quad$ then,

(a) $\quad n \geq (k+1)(\ell+1) + i$

and n can be arbitrarily large.

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(b) $3\left[\dfrac{n-1}{1+k} - 1\right] + 1 \geq d \geq 1$.

(c) $\left[\dfrac{n-1}{\left[\frac{d}{3}\right]+1} - 1\right] \geq k \geq 1$.

Furthermore, all bounds are best possible.

Proof. (a) was proved in Theorem 2.1. To show that n could be as large as desired, take the $H_1(k,d)$ vertex extremal graphs constructed in the proof of Theorem 2.1, then join each of the $n-(k+1(\ell+1)-1$ remaining vertices to the k vertices of V_1 (the notation of the proof of Theorem 2.1 is used here). The resulting graph has, by symmetry, diameter d and its minimal degree is no less than k. From this graph we conclude the following upper bound on $g_1(n,k,d)$ (the lower bound is obvious).

$$\frac{n \cdot k}{2} \leq g_1(n,k,d) \leq g_1(k,d) + (n - f_1(k,d))k .\qquad (2.18)$$

The upper bounds in (b) and (c) are derived from (a) and by the construction in (a) one sees that the bound are best possible. The lower bounds in (b) and (c) are achieved when one takes a complete graph on n vertices (k < n) and a tree with diameter d on n vertices, respectively.

                                                                Q.E.D.

The upper bound in (2.18) will be improved considerably in the following, but first to estimate the "tightness" of the bounds in (2.18) divide the inequality by n and take the limit as $n \to \infty$, as indicated in the introduction.

Since $g_1(k,d)$ and $f_1(k,d)$ do not depend on n we obtain from (2.18)

$$\frac{k}{2} \leq \lim_{n \to \infty} \frac{g_1(n,k,d)}{n} \leq k \; . \tag{2.19}$$

To obtain a better upper bound on $g_1(n,2,d)$, let $n = d+2+(d-1)m+t$, $(0 \leq t \leq d-2)$. Connect two vertices by a path of length $d + 1$, a path of length $t + 1$, and m paths of length d, (Figure 2.9). The resulting graph G
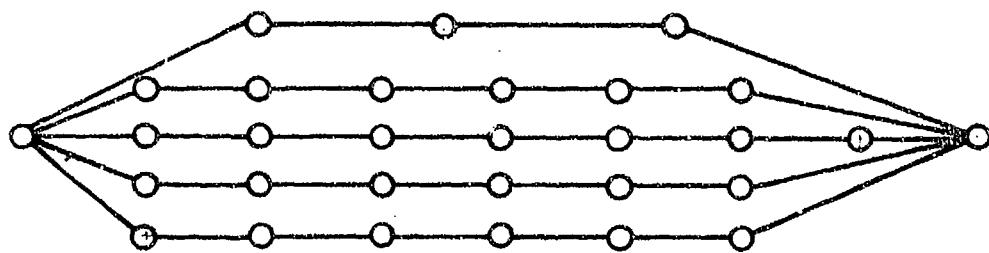
Figure 2.9. An H_1 (30,2,7) – Graph.

has diameter d, n vertices, and its minimal degree is not less than 2. Hence, by counting the edges of the resulting graph, (the lower bound is obvious),

$$n \leq g_1(n,2,d) \leq n + \left[\frac{n-d-2}{d-1}\right] = n - 1 + \left[\frac{n-3}{d-1}\right] \tag{2.20}$$

Applying the tightness measure to (2.20) yields

$$1 \leq \lim_{n \to \infty} \frac{g_1(n,2,d)}{n} \leq 1 + \frac{1}{d-1} \; . \tag{2.21}$$

which for $d \gg 1$ is much better than (2.19).

We present now a method to construct $H_1(n,k,d)$-graphs for $k > 2$ with a "small" number of edges, provided that n is "large" enough. These graphs will improve the upper bound in (2.18).

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First let $d = 2m$, $m \geq 1$. Let $T_{m,k}$ be a hierarchic tree with $m + 1$ levels of vertices, such that there is one vertex of degree one in the first level, then $m - 1$ levels of vertices having degree k and all vertices in the last level have degrees equal to one, (Figure 2.10).
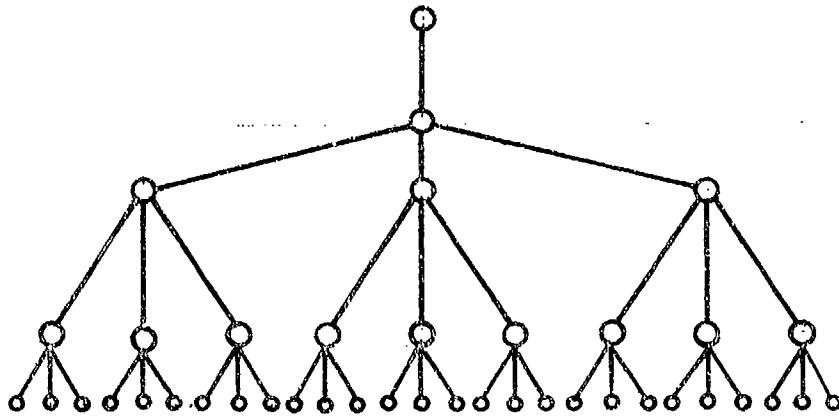
Figure 2.10. $T_{4,4}$.

For $T_{m,k}$ we have,

$$v_{m,k} = \left| V(T_{m,k}) \right| = \frac{(k-1)^m - 1}{k-2} + 1 \; ,$$

$$e_{m,k} = \left| E(T_{m,k}) \right| = \frac{(k-1)^m - 1}{k-2} \; .$$

For given m and k let $n = s(v_{m,k} - 1) + t + 1$, $(0 \leq t < v_{m,k} - 1)$, $s \geq k$. We assume that $n \geq (v_{m,k} - 1) \cdot k + 1$.

If $t = 0$ take s copies of $T_{m,k}$ and identify in each of them the vertex of the first level. The graph obtained is a tree with one vertex of degree $s \geq k$, $s(k-1)^{m-1}$ vertices of degree 1 and all the other vertices have degree k.
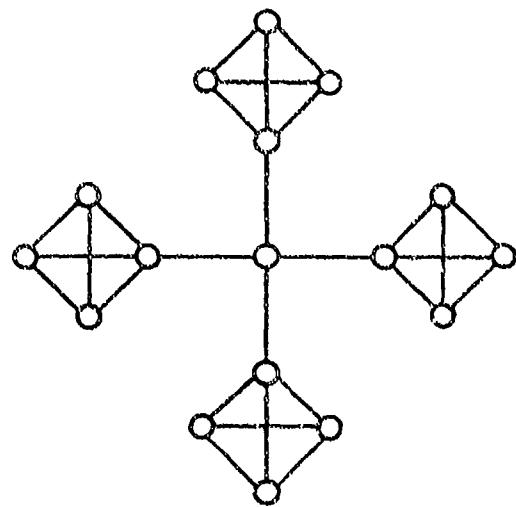
Figure 2.11. Combined $T_{m,k}$'s.

We then complete each group of $k - 1$ vertices of degree 1, belonging
to a common tree $T_{m,k}$, to a complete graph on $k - 1$ vertices, K_{k-1},
(Figure 2.11). Finally, to satisfy the degree requirements join by an
edge the corresponding vertices of successive K_{k-1}'s. If s is even,
the resulting graph is illustrated in Figure 2.12.
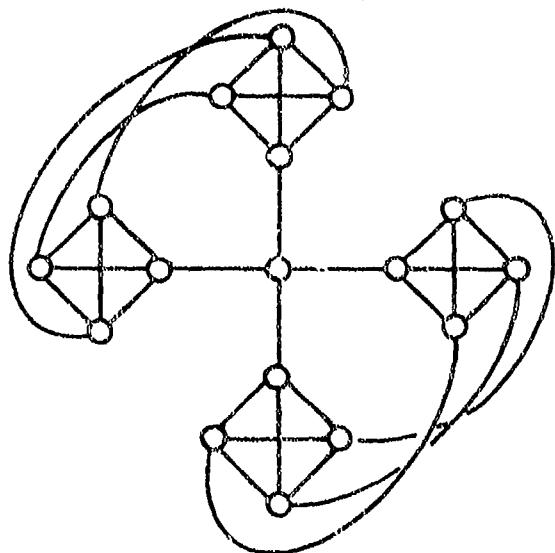
Figure 2.12. A 4-Regular $H_1(17,4,4)$ - Graph.

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For odd s the degrees are attained similarly. The resulting graph has the required minimal degree and is of diameter d.

Note that after combining the s copies of $T_{m,k}$, on the $s(k-1)^{m-1}$ vertices of degree 1 one may also construct a (k-1)-regular or almost (k-1)-regular graph using the methods of [10], (Figure 2.13). This also can be done with the final diameter being exactly d. This construction will have higher connectivity and will be used in the next chapter.
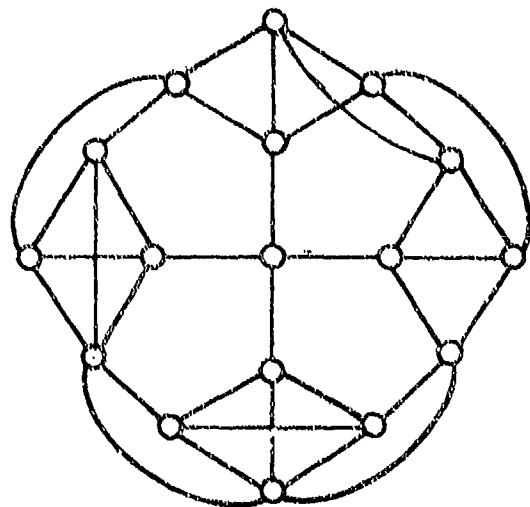
Figure 2.13. A 4-Regular $H_1(17, 4,4)$ – Graph.

If t > 0, construct as before an hierarchic tree on t vertices with vertices of degree k, combine this tree to the previous tree and proceed as before, (Figure 2.14). If only one level or less can be constructed then the graph corresponding to the t vertices is a star.
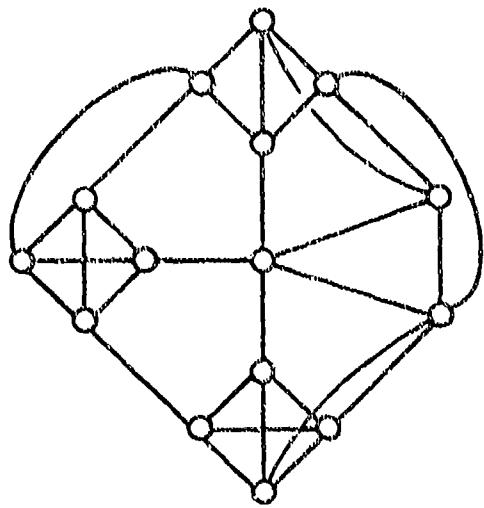
Figure 2.14. An H_1 (15, 4,4) – Graph.

If $d = 2m + 1$, $m \geq 1$, then we take (s-1) hierarchic trees $T_{m,k}$ and one hierarchic tree $T_{m+1,k}$ and proceed as before.

In all cases we are able to construct a graph which has one vertex of degree s and all other vertices have degree k or k + 1. This is possible provided that n is large enough.

By these constructions the upper bound of (2.13) is improved, and is a near optimal value for $g_1(n,k,d)$.

For example, if $d = 2m$, $k = 3$ and $n = (2^m-1)s + 1 + t$, $0 \leq t \leq (2^m-1)$ then, (t = 3) (See Figure 2.15)

$$g_1(n,3,2m) \leq s(2^m+2^{m-1}-1) + 5, \ (s \geq 3) \ . \tag{2.22}$$
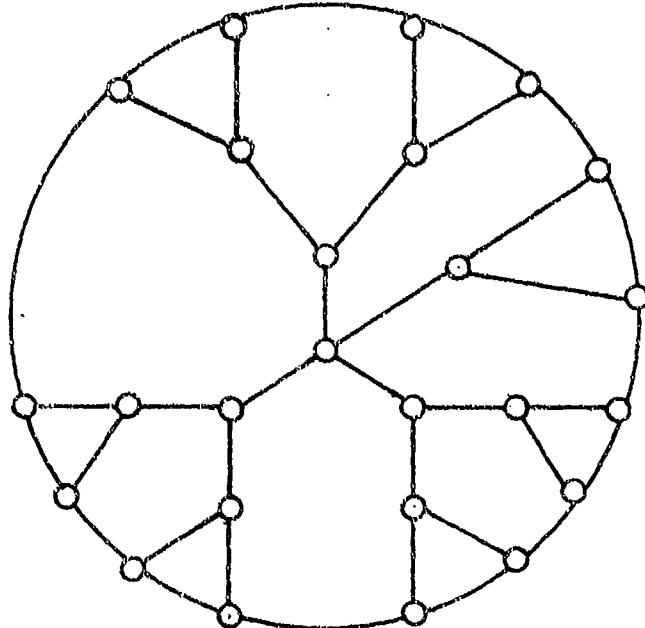
Figure 2.15. An $H_1(25,3,6)$ - Extremal Graph.

If we substitute in (2.22) the value of s and use the obvious lower
bound on $g_1(n,3,2m)$ we obtain, (for $t = 0$),

$$\frac{(n-1)}{2^m-1}\left(2^m + 2^{m-1} - 1\right)+ 5 \geq g_1(n,3,2m) \geq \frac{3}{2}\, n \ . \tag{2.23}$$

The tightness measure on (2.23) yields,

$$\frac{3}{2} + \frac{1}{2}\,\frac{1}{2^m-1} \geq \lim_{n \to \infty} \frac{g_1(n,3,2m)}{n} \geq \frac{3}{2} \ . \tag{2.24}$$

The asumptotic bounds in (2.24) are the same for any n such that
$0 < t < v_{m,k}- 1$, since t is bounded by a function independent of n.

If $m \gg 1$ then (2.24) is a substantial improvement over (2.19),
(when $k = 3$). Similar asymptotic bounds may be obtained for odd d.

In general for $m \geq 3$ from the previous family of graphs that
arises from the trees $T_{m,k}$, $(t = 0)$

$$\frac{nk-k+1+s}{2} \geq g_1(n,k,2m) \geq \frac{k\cdot n}{2}$$

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or after substituting the value of s, $(t = 0)$,

$$\frac{(n-1)k+1}{2} + \frac{(n-1)\ (k-2)}{2((k-1)^m-1)} \geq g_1(n,k,2m) \geq \frac{k \cdot n}{2} \ . \tag{2.25}$$

By applying the tightness measure to (2.25) we obtain,

$$\frac{k}{2} + \frac{1}{2}\ \frac{(k-2)}{(k-1)^m-1} \geq \lim_{n \to \infty} \frac{g_1(n,k,2m)}{n} \geq \frac{k}{2} \ , \tag{2.26}$$

which for $m \gg 1$ improves (2.19). Note that if $t > 0$ we still get (2.26) and that a similar inequality for d odd may be obtained in the same manner. We may therefore summarize,

Theorem 2.5. For $k \geq 3$, $d \geq 2$ and $n \geq (v_{\left[\frac{d}{2}\right],k} - 1) \cdot k + 1$,

$$\frac{k}{2} + \frac{1}{2}\ \frac{k-2}{(k-1)^{\left[\frac{d}{2}\right]}-1} \geq \lim_{n \to \infty} \frac{g_1(n,k,d)}{n} \geq \frac{k}{2} \ .$$

CHAPTER 3

GRAPHS WITH PRESCRIBED DIAMETER AND CONNECTIVITY

3.1 Introduction

Two fundamental considerations in the design of a communication network are its reliability or survivability and its associated maximal transmission delay between any pair of stations. These characteristics depend on the topological configuration of the network. Based on a graph theoretical model of the communication network, many different reliability or survivability criteria may be defined. The simplest criterion used is the minimum number of edges or vertices which must be removed from the graph in order to break all paths between any remaining pair of vertices. Those measures are called the edge connectivity and the vertex connectivity, respectively. In a network where the failure of links is more likely to occur, one uses edge-connectivity as a reliability measure. Whereas in a network whose stations are more likely to fail, vertex connectivity is a more appropriate reliability measure. For given diameter and connectivity values it is generally desirable to construct a network with minimal number of edges. Graphs with given number of vertices and given connectivity, having minimal number of edges, were constructed by F. Harary [10]. However, many of these graphs have a large diameter, and the diameter cannot be prescribed. In this chapter we construct graphs for which the diameter as well as the connectivity are prescribed.

To this end let $H_2(k,d)$ $(H_3(k,d))$ denote the class of k edge connected (k vertex connected) graphs of diameter d. The subclass of graphs in $H_2(k,d)$ $(H_3(k,d))$ having exactly n vertices is denoted by

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$H_2(n,k,d)$ $(H_3(n,k,d))$. For $i = 2,3$ we call the graphs in $H_i(k,d)$ $(H_i(n,k,d))$, $H_i(k,d)$-graphs, $(H_i(n,k,d)$-graphs).

Let

$$f_i(k,d) \triangleq \underset{G \in H_i(k,d)}{\text{Min}} |V(G)| \; ,$$

$$g_i(k,d) \triangleq \underset{G \in H_i(k,d)}{\text{Min}} |E(G)| \; ,$$

$$g_i(n,k,d) \triangleq \underset{G \in H_i(n,k,d)}{\text{Min}} |E(G)| \; .$$

The graphs in these classes with minimal number of vertices are called vertex extremal graphs of the respective classes, and those having minimal number of edges will be called simply extremal graphs. Since a k-connected graph is also k edge connected, and a k-edge connected graph must have a minimal degree larger or equal to k, we have,

$$H_1(k,d) \supset H_2(k,d) \supset H_3(k,d) \tag{3.1}$$

$$H_1(n,k,d) \supset H_2(n,k,d) \supset H_3(n,k,d) \tag{3.2}$$

The extremal graphs constructed in Chapters 2 and 3 show that equality does not hold in (3.1) and (3.2). Some properties of the functions $g_3(k,d)$ and $g_3(n,k,d)$ were studied in [7], [8]. In this chapter the above functions are considered and some extremal graphs are given.

3.2 A Class of $H_3(k,d)$-Extremal Graphs

It is known [12] that,

$$f_3(k,d) = k(d-1) + 2, \qquad (k,d \geq 1), \tag{3.3}$$

and vertex extremal graph are obtained as follows. Let H_1, H_2, ..., H_{d-1}

be disjoint copies of K_k, such that every vertex of H_i is joined by an edge to every vertex of H_{i+1} ($1 \leq i \leq d-2$). Finally join a vertex u to all vertices of H_1 and a vertex v to all vertices of H_{d-1}, (Figure 3.1).
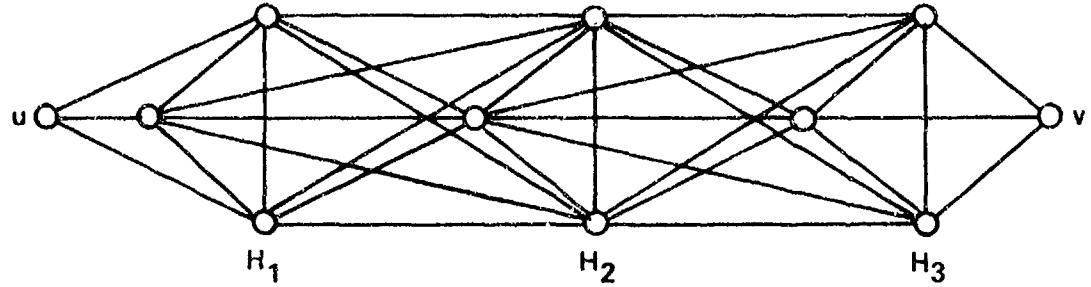
Figure 3.1. A Vertex Extremal H_3 (3,4) - Graph.

Using (3.3) and the fact that every vertex of a k-connected graph has degree k or more, we obtain,

$$g_3(k,d) \geq \left[\frac{f_3(k,d)k+1}{2} \right] = \left[\frac{(k(d-1)+2)k+1}{2} \right] . \qquad (3.4)$$

In (3.4) we have used the fact that the sum of the degrees of the vertices of a graph is always even.

The exact value of $g_3(k,d)$ is calculated in Theorem 3.1.

Theorem 3.1. For any integers $k,d \geq 2$,

$$g_3(k,d) = \left[\frac{(k(d-1)+2)k+1}{2} \right] . \qquad (3.5)$$

Proof. To prove (3.5), it is enough to construct a family of graphs in $H_3(k,d)$ having exactly $\left[\frac{(k(d-1)+2)k+1}{2} \right]$ edges. The construction of these graphs, which we denote by G_k^d will depend on k being even or odd.

Case 1: $k = 2\ell$, $\ell \geq 1$.

Label the k vertices of the complete graph on k vertices, K_k, by $v_1, v_2, \ldots, v_k$. Delete from K_k the edges $v_i v_{i+\ell}$ $(1 \leq i \leq \ell)$, and denote the resulting graph by H (if $\ell = 1$, H is a pair of vertices v_1 and v_j). H is one of the graphs constructed in [10] and shown to be $(k-2)$-connected. Let $H_1, H_2, \ldots, H_{d-1}$ be $d-1$ disjoint copies of H, and denote the vertices of H_j by: $v_{1j}, v_{2j}, \ldots, v_{kj}$ $(1 \leq j \leq d-1)$. Form $G_{2\ell}^d$ in the following way. Join v_{i_j} to $v_{i_{j+1}}$ by an edge for all $1 \leq i \leq k$ and $1 \leq j \leq d-2$. Then join a new vertex u_1 adjacent to all the vertices of H_1, and a vertex u_2 adjacent to all the vertices of H_{d-1}, (Fig. 3.2).
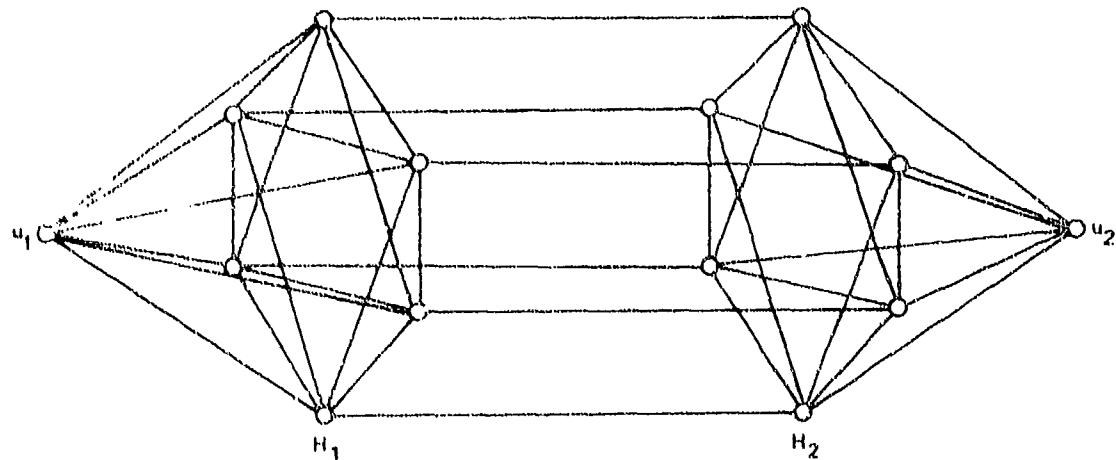
Figure 3.2. G_8^3.

Counting the edges we obtain

$$|E(G_{2\ell}^d)| = (\ell(d-1)+1)2\ell .$$

(3.6)

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<u>Case 2</u>: $k = 2\ell + 1$, $\ell \geq 1$.

We begin by drawing a k-cycle C_k, with vertices $v_1, v_2, \ldots, v_k$. Then we join 2 vertices v_i and v_j if and only if $|i-j| \equiv m \pmod{k}$, where $2 \leq m \leq \ell - 1$ (if $\ell = 2$, we obtain a 5-cycle). Finally join v_i to $v_{i+\ell}$ by an edge for $1 \leq i \leq \ell$ (see [10]). The resulting graph is again denoted by H (if $\ell = 1$, H is a disconnected graph composed of an edge $v_1 v_2$ and an isolated vertex v_3). We have, in H,

$$\deg(v_i) = k - 2, \qquad 1 \leq i \leq 2\ell,$$
$$dg(v_{2\ell+1}) = k - 3.$$

Let $H_1, H_2, \ldots, H_{d-1}$ be d-1 disjoint copies of H. Denote the vertices of H_j by $v_{1_j}, v_{2_j}, \ldots, v_{k_j}$, $1 \leq j \leq d-1$, in such a way that

$$\deg(v_{1_s}) = \deg(v_{2_{s+1}}) = k - 3 \qquad (s\text{-odd}) .$$

To construct $H_3(2\ell+1, d)$-graphs, two subcases are considered.

(I) $d = 2n+1$, $n \geq 1$.

To form $G_{2\ell+1}^{2n+1}$ join v_{i_j} to $v_{i_{j+1}}$ by an edge for all $1 \leq i \leq k$ and $1 \leq j \leq d$. Then join new vertices u_1 and u_2 to all the vertices of H_1 and H_{d-1}, respectively and finally join v_{1_s} to $v_{2_{s+1}}$ for s odd $(1 \leq s \leq d-1)$, (Figure 3.3). We then have

$$\left| E\left(G_{2\ell+1}^{2n+1}\right) \right| = (n(2\ell+1)+1)(2\ell+1) . \tag{3.7}$$
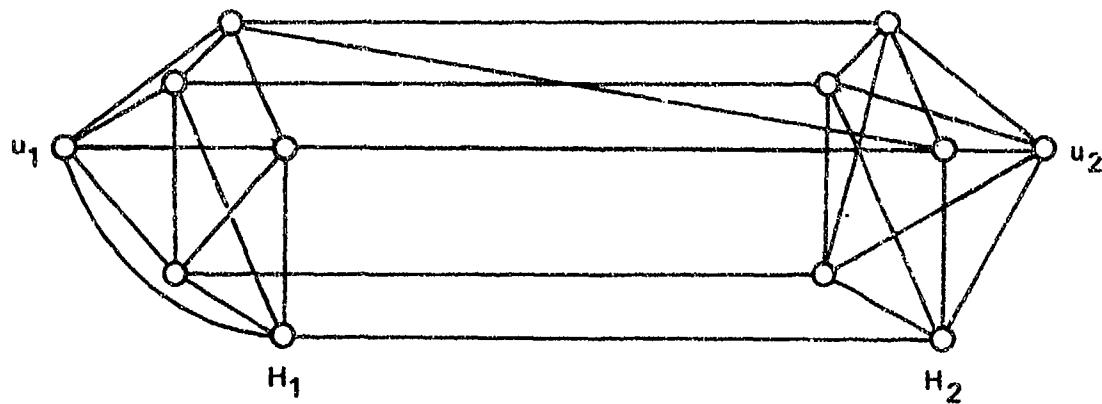
Figure 3.3. G_5^3.

(II) $d = 2n$, $n \geq 1$.

The construction of $G_{2\ell+1}^{2n}$ is similar to the construction in (I),
except that v_{1_s} is joined to $v_{2_{s+1}}$ for s odd only for $1 \leq s \leq d-3$, and
$v_{1_{d-1}}$ is connected by an edge to $v_{1+\ell_{d-1}}$, (Figure 3.4).
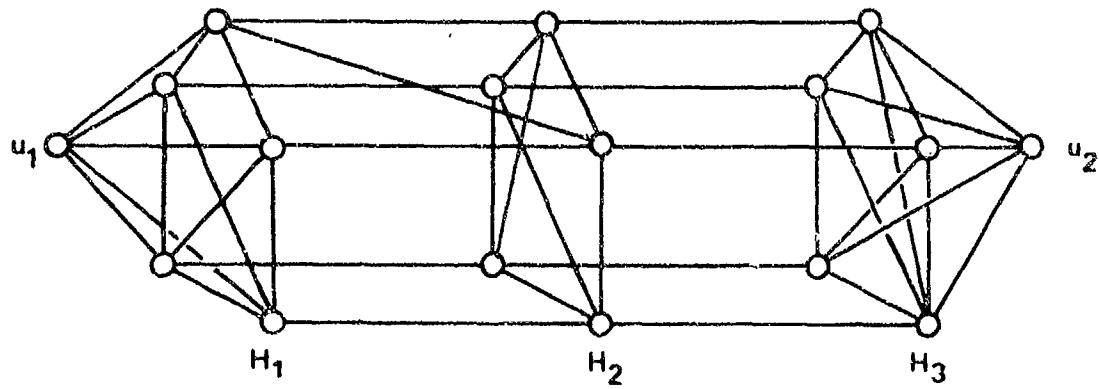
Figure 3.4. G_5^4.

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By counting the edges we obtain,

$$|E\left(G_{2\ell+1}^{2n}\right)| = \frac{((2\ell+1)(2n-1)+2)(2\ell+1)+1}{2} .\tag{3.8}$$

summarizing (3.6), (3.7) and (3.8) we conclude,

$$|E\left(G_k^d\right)| = \left[\frac{(k(d-1)+2)k+1}{2}\right] ,\tag{3.9}$$

for all $k,d \geq 2$.

Each pair of vertices in G_k^d ($k,d \geq 2$) is contained in a cycle of length $\leq 2d$, therefore, $d\left(G_k^d\right) \leq d$. Since $d(u_1, u_2) = d$, we conclude,

$$d\left(G_k^d\right) = d.$$

To complete the proof, the k-connectivity of G_k^d must be established. To that end we state the following definition and assertion.

We shall say that a vertex $v_{i_j} \in H_j$ has the star property if v_{i_j} is adjacent to all vertices of H_j except one vertex, say $v_{i_{o_j}}$.

Assertion: If the vertices $v_{i_\ell} \in H_\ell$ and $v_{j_n} \in H_n$ ($1 \leq \ell < n \leq d-1$), both have the star property, then there are k-vertex disjoint v_{i_ℓ}, v_{j_n}-paths in G_k^d.

To prove the assertion consider the following paths,

$$v_{i_\ell} v_{m_\ell} v_{m_n} v_{j_n} \qquad 1 \leq m, k \qquad m \neq i_o, j_o .$$

$$v_{i_\ell} u_1 v_{i_{o_\ell}} v_{i_{o_n}} v_{j_n}$$

$$v_{i_\ell} v_{j_{o_\ell}} v_{j_{o_n}} u_2 v_{j_n}$$

Note that i may be equal to j, i to m etc. Since the above k v_{i_ℓ}, v_{j_n}-paths are vertex disjoint, the assertion is proved.

Let a, $b \in V\left(G_k^d\right)$, if $\{a,b\} = \{u_1, u_2\}$ then clearly there are k vertex disjoint a,b-paths in G_k^d. Otherwise, a few cases and subcases need to be considered.

Case 1: $k = 2\ell$, $\ell \geq 1$.

I. $a = u_1$, $b \in H_j$ (and the symmetric case).

Since b has the star property, one easily asserts the existence of the required paths.

II. a, $b \in H_i$.

H_i is $(k-2)$-connected and therefore there are $k-2$ vertex disjoint a,b-paths in H_i. Furthermore, there is a cycle through a, b, u_1 and u_2 which has only a and b in common with the previous $k-2$ a,b-paths.

III. $a \in H_i$, $b \in H_j$ and $i \neq j$.

Since both a and b have the star property, the existence of the vertex disjoint a,b-paths follows by the assertion.

By the Menger-Whitney Theorem, $G_{2\ell}^d$ is k-connected.

Case 2: $k = 2\ell + 1$, $\ell \geq 1$

Consider only the case d odd (d even is treated similarly)

I. $a = u_1$, $b \in H_j$ (and the symmetric case). If b has the star property in H_j then the paths exist like in Case 1 (I). Otherwise, $b = v_{1_s}$ or $b = v_{2_{s+1}}$ for some $1 \leq s \leq d-1$, and again the existence of the paths is easily observed.

II and III. a, $b \in H_i$ and a does not have the star property in H_i (there is only a single such vertex in H_i). Or $a \in H_i$, $b \in H_j$, and a or b or both do not have the star property. Then in a similar manner, we establish the k disjoint a,b-paths and the k-connectivity follows.

Q.E.D.

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It should be noted that the $H_3(k,d)$-extremal graphs obtained in this section, are $H_1(n,k,d)$-extremal graphs, where n has the appropriate value.

3.3 Inequalities Concerning the Parameters of $H_3(n,k,d)$

Considering a k-connected graphs with n vertices and diameter d, we derive a few relationships between the parameters n, k and d.

Theorem 3.2. For all $H_3(n,k,d)$-graphs

(a) $n \geq (d-1)k + 2$ (3.10)

and n can be as large as desired.

(b)   If $k = n - 1$, then $d = 1$

If $k < n - 1$ then

$$\left[\frac{n-2}{k} + 1\right] \geq d \geq 2 \ .$$ (3.11)

(c) $\left[\frac{n-2}{d-1}\right] \geq k \geq 1,$ (3.12)

and all the bounds are best possible.

Proof.   (a) As mentioned before, (3.10) is proved in [12].

To construct a graph $G \in H_3(n,k,d)$, with any given n, provided that $n \geq (d-1)k + 2$, take the $H_3((d-1)k + 2, k,d)$-extremal graph as in the proof of Theorem 3.1.   Then add the remaining $n-(d-1)k-2$ vertices, and join each of them to all the k vertices of H_1 by an edge, (the notation of the proof of Theorem 3.1 is used here).   The resulting graph (Figure 3.5) has, by symmetry, diameter d and is k-connected, therefore, it is an $H_3(n,k,d)$-graph.   The latter can be used to establish an upper bound on $g_3(n,k,d)$,

$$\frac{k}{2} (2n-kd+k-2) \geq g_3(n,k,d) \geq \frac{nk}{2} \ .$$ (3.13)

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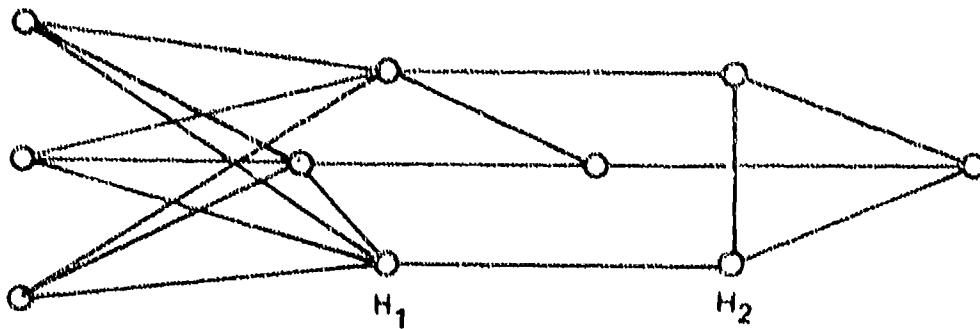
Figure 3.5. An H_3 (10,3,3) -- Graph.

(b) If $G \in H_3(n,k,d)$ and $k=n-1$ then, $|E(G)| \geq \frac{(n-1)n}{2}$. Hence, $G=k_n$.

Otherwise, $k < n-1$ and $d \geq 2$. Using Harary's method [10], a $(k-1)$-

connected graph on $n-1$ vertices may be constructed. By adding a new

vertex adjacent to all previous $n-1$ vertices an $H_3(n,k,2)$-graph is

obtained, which proves that the lower bound in (3.11) achieved.

The upper bounds in (3.11), (3.12) are derived from (3.10) and the

graph constructed in part (a) attains them.

(c) To show that for any n and d these exists a 1-connected graph

with n vertices and diameter d, simply take a path of length d and join

all the other $n > d + 1$ vertices to any vertex of degree two on the

path.

Q.E.D.

3.4 Bounds on $g_3(n,k,d)$

Clearly, any k-regular graph in $H_3(n,k,d)$ will be $H_3(n,k,d)$-

extremal. The following theorem gives the construction of such a class

of graphs for k even, for a limited range of n.

Theorem 3.3. For $\ell \geq 2$ and

$$(d-1)2\ell+2 \leq n \leq (d-1)(\ell-1)4+2,$$

$$g_3(n,2\ell,d) = n\ell .$$
(3.14)

Proof. Since $g_3(n,2\ell,d) \geq n\ell$, any construction of 2ℓ-regular graphs in $H_1(n,2\ell,d)$, for $(d-1)2\ell+2 \leq n \leq (d-1)(\ell-1)4+2$ will prove (3.14)

If $n = (d-1)2\ell+2$, the extremal graph was constructed in the proof of Theorem 3.1. Assume $n-(d-1)2\ell-2 = m > 0$, and start with the $H_3((d-1)2\ell+2, 2\ell, d)$-extremal graph of Theorem 3.1. Add the other m vertices $w_1, w_2, \ldots, w_m$, $(1 \leq m \leq (d-1)(2\ell-4))$ to the extremal graph as follows. If $m \leq 2\ell - 4$ join each of $w_i (1 \leq i \leq m)$ to all vertices of H_1, and change H_1, which is a $(2\ell-2)$-regular, $(2\ell-2)$-connected graph on 2ℓ vertices, to a $(2\ell-2-m)$-regular $(2\ell-2-m)$-connected graph on the 2ℓ vertices of H_1, using the methods of [10]. If $m > 2\ell - 4$ connect the remaining vertices to H_2, while reducing the regularity and connectivity of H_2 appropriately, then do the same to H_3 etc. (Figure 3.6). The total number of vertices one may add to the extremal graph in this way is $(2\ell-4)(d-1)$. It is easy to verify the resulting graph is 2ℓ-regular 2ℓ-connected and has diameter d.

Q.E.D.
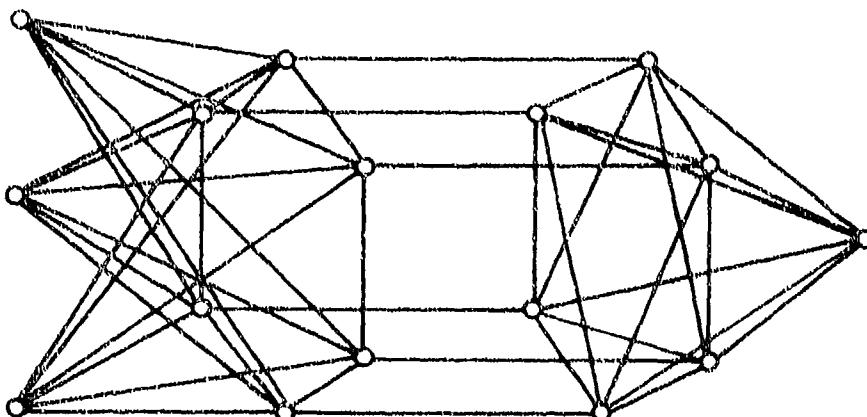
Figure 3.6. An H_3 (16,6,3) — Extremal Graph.

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Note that this method of modifying the extremal graph can be applied to graphs in $H_3(n,k,d)$ with k odd and restricted to appropriate values of n and hence obtain a class of $H_3(n,k,d)$-extremal graphs for k odd, (which will be k-regular or almost k regular). Further, it can also be applied to obtain $H_1(n,k,d)$-extremal graphs mentioned in Chapter 2.

If $n > (d-1)(\ell-1)4+2$, an upper bound on $g_3(n,2\ell,d)$ may be obtained by adding new vertices to the $H_3((d-1)(\ell-1)4+2, 2\ell, d)$-extremal graph, and joining each of them by an edge to all vertices of H_1. Thus an improvement of (3.13) is obtained (the lower bound is obvious),

$$n\ell \leq g_3(n,2\ell,d) \leq \ell(2n-(d-1)(4\ell-4)-2) . \qquad (3.15)$$

Applying the tightness measure to (3.15) we obtain,

$$\ell \leq \lim_{n \to \infty} \frac{g_3(n,2\ell,d)}{n} \leq 2\ell , \qquad (3.16)$$

which is similar to (2.19).

It should be noted that some of the graphs constructed in Section 2.4 of Chapter 2 (Figures 2.9-2.13), which were of diameter d, minimal degree k and having exactly n vertices, are also k-connected and therefore some of the constructions may be used to obtain $H_3(n,k,d)$ and $H_3(n,k,d)$-extremal graphs.

Since the $H_1(n,2,d)$-graph constructed in Section 2.4, (See Figure 2.9), is clearly 2-connected, inequalities (2.20) and (2.21) are true for $g_3(n,2,d)$ also.

Hence,

$$n \leq g_3(n,2,d) \leq n + \left[\frac{n-d-2}{d-1}\right] , \qquad (3.17)$$

and,

$$1 \leq \lim_{n \to \infty} \frac{g_3(n,2,d)}{n} \leq 1 + \frac{1}{d-1} \quad . \tag{3.18}$$

Since the $H_1(n,3,d)$-graph obtained in Section 2.4 (Fig. 2.15) is also 3-connected we have the equivalent of (2.24) for $g_3(n,3,d)$ as follows,

$$\frac{3}{2} \leq \lim_{n \to \infty} \frac{g_3(n,3,d)}{n} \leq \frac{3}{2} + \frac{1}{2} \frac{1}{2^{\left\lfloor \frac{d}{2} \right\rfloor -1}} \quad . \tag{3.19}$$

For $k > 3$ take the s copies of $T_{m,k}$ already combined as described in Section 2.4 and on the $s(k-1)^{m-1}$ vertices of degree construct a $(k-1)$-regular, $(k-1)$-connected, or almost $(k-1)$-regular, $(k-1)$-connected graph by the methods of Harary [10] as illustrated in Figures 2.13 and 2.14. The resulting graph is obviously k-connected on n vertices and has diameter $\leq d$. Therefore if we denote the class of k-connected graph on n vertices with diameter $\leq d$ by $\overline{H}_3(n,k,d)$, $(\overline{H}_3(n,k,d) \supset H_3(n,k,d))$ and the minimal number of edges of an $\overline{H}_3(n,k,d)$-graph by $\overline{g}_3(n,k,d)$ then, for $t = 0$ we obtain from (2.25),

$$\frac{k \cdot n}{2} \leq \overline{g}_3(n,k,2m) \leq \frac{(n-1)k+1}{2} + \frac{(n-1)(k-2)}{2((k-1)^m-1)} \quad . \tag{3.20}$$

Corresponding to Theorem 2.5 we obtain for $k \geq 3$, $d \geq 2$ and $n \geq (v_{\left\lceil \frac{d}{2} \right\rceil, k} - 1)k+1$

$$\frac{k}{2} \leq \lim_{n \to \infty} \frac{\overline{g}_3(n,k,d)}{n} \leq \frac{k}{2} + \frac{1}{2} \frac{k-2}{(k-1)^{\left\lceil \frac{d}{2} \right\rceil -1}} \quad . \tag{3.21}$$

3.5 Remark on Graphs with Prescribed Edge Connectivity

Considering the synthesis of reliable communication networks with respect to link failures, the following problem is of interest. Given integers d, k $\geq$ 2, we wish to find the minimal number of vertices and edges that an k-edge connected graph with diameter d must have. Fulkerson and Shapley [11] solved the problem of finding the minimal number of edges that an k-edge connected graph on n vertices must have, but in their work the diameter is not prescribed. We, on the other hand, will use both edge connectivity and diameter constraints.

In terms of the definitions made in Section 3.1, we will obtain bounds on the functions $f_2(k,d)$, $g_2(k,d)$ and $g_2(n,k,d)$. First note that due to (3.1) we have for integers k, d $\geq$ 2,

$$f_1(k,d) \leq f_2(k,d) \leq f_3(k,d) \tag{3.22}$$

$$g_1(k,d) \leq g_2(k,d) \leq g_3(k,d) . \tag{3.23}$$

For n $\geq$ (d-1)k+2 we also have from (3.2),

$$g_1(n,k,d) \leq g_2(n,k,d) \leq g_3(n,k,d) . \tag{3.24}$$

However, one can obtain tighter bounds on $f_2(k,d)$ and $g_2(k,d)$ as follows. Consider the class of $H_3(k,d)$-extremal graphs constructed in the proof of Theorem 3.1 (Figures 3.2, 3.3 and 3.4), using the same notation as in the theorem. Instead of H_1, H_3, H_5,...,$H_{2\ell+1}$, as defined in Section 3.2, we take single vertices. The graphs obtained in this way are clearly of diameter d and are k-edge connected, thus being $H_2(k,d)$-graphs (Figure 3.7, 3.8). By counting the number of vertices of such graphs, we obtain an upper bound on $f_2(k,d)$.

$$f_1(k,d) \le f_2(k,d) \le \begin{cases} \dfrac{(k+1)d}{2} + 1 & , \quad d\text{-even} \\[3mm] \dfrac{(k+1)(d+1)}{2} & , \quad d\text{-odd} . \end{cases} \tag{3.25}$$
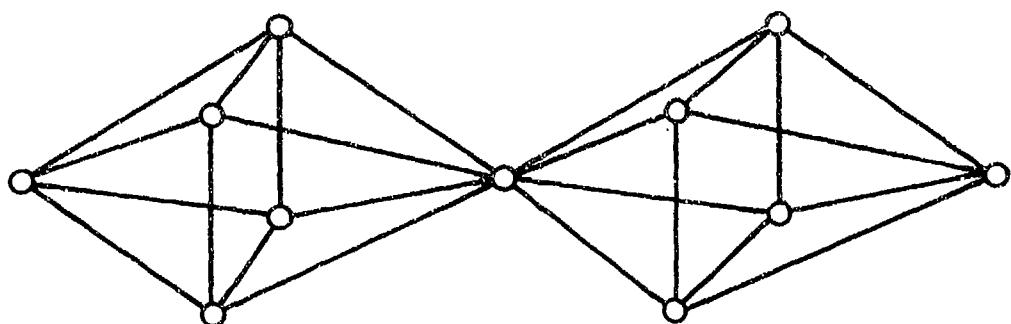
Figure 3.7. An H_2 (4,4) — Graph.
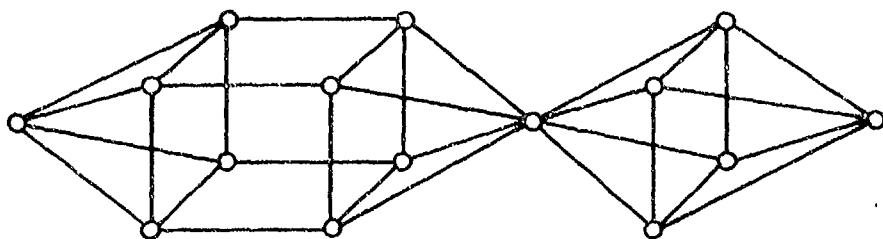
Figure 3.8. An H_2 (4,5) — Graph.

Furthermore, (using the notation of Section 2.2) an $H_2(k,d)$-graph must have at least k edges between the vertices of V_i and V_{i+1}, $0 \le i \le d-1$. Otherwise, the graph cannot be k-edge connected.

Hence, we must have,

$$n_i \cdot n_{i+1} \ge k \quad , \quad 0 \le i \le d-1 . \tag{3.26}$$

From (3.26) we obtain

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$$n_i + n_{i+1} \geq 2\sqrt{k} \quad , \quad 0 \leq i \leq d - 1 . \tag{3.27}$$

Inequality (3.27) follows by noting that if we assume,

$$n_i + n_{i+1} < 2\sqrt{k} ,$$

together with using inequality $n_{i+1} \geq \dfrac{k}{n_i}$, we obtain $n_i + \dfrac{k}{n_i} < 2\sqrt{k}$, or $n_i^2 + k < 2\sqrt{k} \, n_i$, or $(n_i - k)^2 < 0$, which is a contradiction. Incorporating inequalities (2.2) which must be satisfied by any $H_2(k,d)$-graph, we have for $d \geq 3$,

$$\sum_{i=2}^{d-2} n_i \geq 2(k+1) + (d-3)\sqrt{k}.$$

Combining the latter inequality with the upper bounds in (3.25) we obtain

$$\underset{k,d \geq 2}{\text{Max}} \left\{ f_1(k,d), 2(k+1)+(d-3)\sqrt{k} \right\} \leq f_2(k,d) \leq \begin{cases} \dfrac{(k+1)d}{2} + 1 \, , \ d\text{-even} \\ \\ \dfrac{(k+1)(d+1)}{2} \, , \ d\text{-odd} . \end{cases} \tag{3.28}$$

In some special cases the exact value of $f_2(k,d)$ can be obtained. In other cases, tighter bounds than in (3.28) may be obtained, but the general formula for $f_2(k,d)$ seems to be much more difficult to get than the formula for $f_1(k,d)$ and $f_3(k,d)$.

We list a few special cases.

(1) $f_2(k,1) = k + 1, \ f_2(k,2) = k + 2,$

these are readily proved.

(2) For $k = 4$, $d \geq 6$, the following graph attains the lower bound in (3.28) (Figure 3.9).
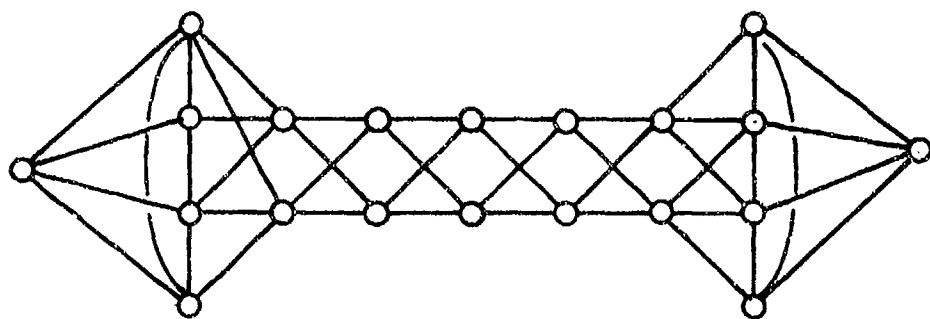
Figure 3.9. A Vertex-Extremal H_2 (4,8) — Graph.

Thus, for $d \geq 6$,

$$f_2(4,d) = 2d + 4 .$$ (3.29)

(3) For $k = 5$, $d \geq 6$, consider Figure 3.10.
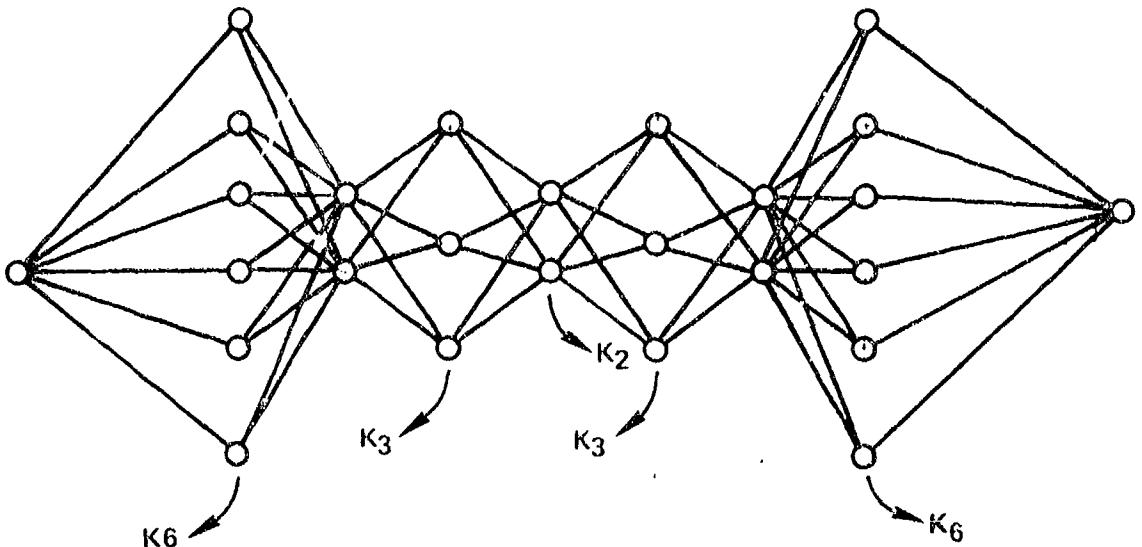
Figure 3.10. An H_2 (6,8) — Graph.

If $d \geq 8$, the construction in Figure 3.10 yields a smaller upper bound than in (3.28). For this special graph we obtain,

$$24 \leq f_2(6.8) \leq 26 ,$$

where the lower bound is derived from (3.28) and the upper bound from Figure 3.10.

By counting the number of edges of the previously constructed graphs (Figures 3.7 and 3.8), upper bounds on $g_2(k,d)$ are obtained. By imposing the existence of at least k edges between V_i and V_{i+1} $0 \leq i \leq d-1$, lower bound are attained.

Thus, we find

$$dk \leq g_2(k,d) \leq \begin{cases} \dfrac{dk(k+2)}{4} & \text{d-even, k-even} \\[2ex] \dfrac{k(k(d+1)+2(d-1))}{4} & \text{d-odd, k-even} \\[2ex] \dfrac{(k+1)(4k+(d-3)(k+1))}{4} & \text{d-odd, k-odd} \\[2ex] \dfrac{d(k+1)^2}{4} & \text{d-even, k-odd .} \end{cases} \tag{3.30}$$

As in calculating bounds on $f_2(k,d)$, the upper bounds on $g_2(k,d)$ can be improved using the above mentioned techniques.

Noting that the construction in Section 2.4, used to obtain an upper bound on $g_1(n,2,d)$ (Figure 2.9), is clearly an $H_2(n,k,d)$-graph we obtain,

$$n \leq g_2(n,2,d) \leq n + \left[\frac{n-d-2}{d-1} \right] . \tag{3.31}$$

Therefore,

$$1 \leq \lim_{n \to \infty} \frac{g_2(n,2,d)}{n} \leq 1 + \frac{1}{d-1} . \tag{3.32}$$

Similarly, from the $H_1(n,3,d)$-graphs in Section 2.4 (Figure 2.15), which are also $H_2(n,3,d)$-graphs, we may write,

$$\frac{3}{2} \leq \lim_{n \to \infty} \frac{g_2(n,3,d)}{n} \leq \frac{3}{2} + \frac{1}{2} \frac{1}{2^{\left\lfloor \frac{d}{2} \right\rfloor - 1}} . \tag{3.33}$$

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For $k > 3$, we take the s copies of $T_{m,k}$ and combine them as in Section 2.4. On the $s(k-1)^{m-1}$ vertices of degree 1 of the resulting graph, we construct a $(k-1)$-regular, or almost $(k-1)$-regular, $(k-1)$-connected graph, by the methods of [10] or [11] to obtain a k-edge connected graph on n vertices with diameter $\leq d$. Therefore, if $\overline{H}_2(n,k,d)$ denotes the class of k-edge connected graphs on n vertices with diameter $\leq d$, and the minimal number of edges of an $\overline{H}_2(n,k,d)$-graph is denoted by $\overline{g}_2(n,k,d)$, we obtain using the graphs thus constructed, $(k > 3)$

$$\frac{k}{2} \leq \lim_{n \to \infty} \frac{\overline{g}_2(n,k,d)}{n} \leq \frac{k}{2} + \frac{1}{2} \frac{k-2}{(k-1)^{\left\lceil \frac{d}{2} \right\rceil - 1}} \quad . \tag{3.34}$$

CHAPTER 4

ON THE DIAMETER STABILITY OF GRAPHS

4.1 Introduction

We consider a communication network such as a store and forward
message switching computer communication network. The network is topo-
logically described by an underlying graph whose vertices represent the
network terminals and switches and whose edges represent the network
communication channels. Messages arriving at random at the network
source terminals and routed through the network towards the correspond-
ing destination terminals, experience queueing and transmission time
delays. When we use the maximal average message delay as the network
delay measure, the diameter of the underlying graph can be shown to
serve as an index of the network message delay performance (see [1]-[2]).
The use of connectivity indices of the graph as a reliability measure is
based on the assumption that the network is operational in the presence
of failure provided there is at least one path remaining between every
pair of nodes. However under failures the resulting network may have an
excessively large diameter, which as previously indicated may result in
intolerable queueing delays while routing a message through the network.
Therefore, a more meaningful reliability measure for a computer network
would be the minimum number of nodes or links that must fail in order for
the diameter of the graph to exceed a specified value. (See also Wilkov
[28]).

In this chapter we assume that it is required to construct the net-
work in such a way that the maximal message delay will not exceed a pre-
scribed upper bound, even if a number of communication links fail.

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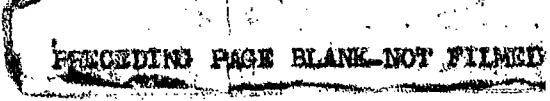
Hence the following graph theoretical parameters are defined and investigated.

A graph G is defined to be an (ℓ,d)-graph (with respect to edges) if

$$d(G-E) \le d,$$

$\forall E \subseteq E(G)$ such that $|E| \le \ell-1$. Or equivalently a graph G is an (ℓ,d)-graph if and only if for any two distinct vertices x,y and any $\ell-1$ disjoint edges e_1, e_2, ..., $e_{\ell-1}$, there exists an x,y-path of length not exceeding d, which avoids e_1, e_2, ..., $e_{\ell-1}$.

To show the equivalence of the definitions let G be an (ℓ,d)-graph, then by removing e_1, e_2, ..., $e_{\ell-1}$ from G the diameter of the resulting graph, $G - \{e_1$, e_2, ..., $e_{\ell-1}\}$, does not exceed d, consequently for any pair of distinct vertices x,y there is an x,y-path of length $\le d$ avoiding e_1, e_2, ..., $e_{\ell-1}$. To show the sufficiency of the condition, let a_1, a_2, ..., a_k $(k < \ell)$ be a minimal set of edges of G whose elimination results in a graph of diameter $> d$. Therefore, there exists $x,y \in V(G)$ such that $d_{G-A}(x,y) > d$, where $A = \{a_1$, a_2, ..., $a_k\}$. Choose $\ell-k-1$ other distinct edges a_{k+1}, a_{k+2}, ..., $a_{\ell-1}$. According to the condition in the definition there exists an x,y-path of length $\le d$, avoiding a_1, a_2, ..., $a_{\ell-1}$ contrary to our assumption. Hence G is an (ℓ,d)-graph.

(ℓ,d)-stable graphs (with respect to vertices) are defined similarly, as graphs G with the property

$$d(G-V) \le d,$$

$\forall V \subseteq V(G)$ such that $|V| \le \ell-1$.

By removing a vertex $v \in V(G)$ from G, yielding $G - \{v\}$, we mean removing v and all edges incident with v. Since only (ℓ,d)-graph with respect to edges are investigated in this chapter, we call them simply (ℓ,d)-graphs.

Clearly, $d \geq d(G)$. An (ℓ,d)-graph with diameter d is called an ℓ-diameter stable graph. Note that an (ℓ,d)-graph is well defined only if ℓ, $d \geq 2$ and that the diameter of an ℓ-diameter stable graph must be at least 2.

A pair of nonadjacent vertices $x,y \in V(G)$ will be called an ℓ-distance pair if

$$d_G(x,y) = d_{G-E}(x,y),$$

$\forall E \subset E(G)$ such that $|E| \leq \ell-1$.

A graph is called ℓ-distance stable if and only if all pairs of nonadjacent vertices of the graph are ℓ-distance pairs. In an ℓ-diameter stable graph one has to remove at least ℓ edges from the graph in order to increase its diameter. Whereas in an ℓ-distance stable graph at least ℓ edges must be removed from the graph in order to increase the distance between any pair of nonadjacent vertices of the graph. (ℓ,d)-graph are in particular ℓ edge connected graphs and the following relations between the respective classes exist.

$\{\ell$-edge connected graphs$\} \supset \{(\ell,d)$-graphs$\} \supset \{\ell$-diameter stable graph$\} \supset \{\ell$-distance stable graphs$\}$.

The above inclusions are sharp as will be shown in the examples of Section 4.2. It is easy to find graphs that are ℓ-edge connected and not (ℓ,d)-graphs, etc.

If d is "large" then the class of (ℓ,d)-graphs is identical with the class of ℓ-edge connected graphs.

The above definition of stability with some results appeared in [5] and [6]. Some properties of ℓ-distance stable graphs (mainly with respect to vertices) were obtained independently in [23], where they are called ℓ-geodetically line (vertex)-connected graphs. Some examples of (ℓ,d)-graphs may be found in [15]-[22], where the main problem is to find the minimal number of edges that an (ℓ,d)-graph with diameter δ and n vertices must have and to construct classes of extremal graphs in this sense. Later in the chapter we will refer to those works. In the following, some properties of (ℓ,d)-graphs, ℓ-diameter stable graphs and ℓ-distance stable graphs are derived and some related extremal problems are investigated.

4.2 Examples of (ℓ,d)-graphs

Example 4.2.1

A class of (ℓ,d)-graphs having n vertices may be obtained by a simple application of a result obtained by J. W. Moon in [24]. There the function $g(n,d)$ $(n-1 > d > 2)$ is defined as the least integer r such that if the degree of every vertex of G $(|V(G)| = n)$ is greater or equal to r, then $d(G) \leq d$. Moon obtained

$$g(n,d) = \begin{cases} \left\lceil \dfrac{n}{t} \right\rceil & \text{if } d = 3t - 4 \\[2mm] \left\lceil \dfrac{n-1}{t} \right\rceil & \text{if } d = 3t - 3 \\[2mm] \left\lceil \dfrac{n-2}{t} \right\rceil & \text{if } d = 3t - 2. \end{cases}$$

Therefore, if all vertices of G satisfy

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$$\deg(v) \geq \begin{cases} \left\lceil \dfrac{n}{t} \right\rceil + \ell - 1 & \text{if } d = 3t - 4 \\[2mm] \left\lceil \dfrac{n-1}{t} \right\rceil + \ell - 1 & \text{if } d = 3t - 3 \\[2mm] \left\lceil \dfrac{n-2}{t} \right\rceil + \ell - 1 & \text{if } d = 3t - 2, \end{cases}$$

then G is an (ℓ,d)-graph. Note that if the number of vertices is pre-scribed, an (ℓ,d)-graph can be obtained by using the methods of Chapter 2 and 3. However, the latter graphs will have many extra edges.

Example 4.2.2.

Consider a collection of $(2m+1)$-cycles all sharing exactly one vertex and let $m \leq \dfrac{d}{3}$. The graph obtained is $(2,d)$-stable.
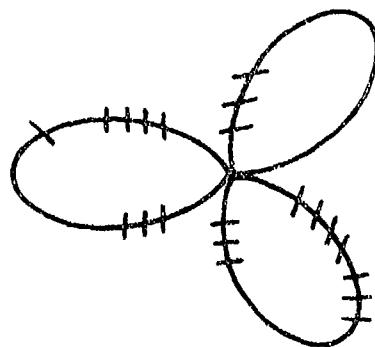
Figure 4.1. A (2,d) – Stable Graph.

Note that this graph is not a 2-diameter stable graph.

We now state three lemmas that will be used for constructing (ℓ,d)-graphs. The proofs are obvious and therefore omitted.

Lemma 4.1. If between any pair of vertices of G there are at least ℓ edge disjoint paths not longer than d, then G is an (ℓ,d)-graph.

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Lemma 4.2. If G_1 and G_2 are ℓ-diameter stable graphs and $x_i, y_i \in V(G_i)$ is a diametrical pair of G_i $(i=1,2)$, then the graph G obtained by identifying the vertex $x_1 \in V(G_1)$ with $x_2 \in V(G_2)$ is an ℓ-diameter stable graph of diameter $d(G_1) + d(G_2)$.

Lemma 4.3. If for any pair of nonadjacent vertices $x, y \in V(G)$, there are at least ℓ edge disjoint x,y-paths of length $d_G(x,y)$, then G is an ℓ-distance stable graph.

Example 4.2.3.

Let $\ell, d \geq 2$ be any arbitrary integers. Let H_1, H_2, ..., H_{d-1} be $d-1$ disjoint copies of the complete ℓ-vertex graph K_ℓ, where the vertices of H_j are denoted by v_{1_j}, v_{2_j}, ..., v_{ℓ_j} $(1 \leq j \leq d-1)$. Join v_{i_j} to $v_{i_{j+1}}$ by an edge for all $1 \leq i \leq \ell$ and $1 \leq j \leq d-1$. Then join a new vertex u, adjacent to all vertices of H_1, and a vertex u, adjacent to all vertices of H_{d-1}, (Fig. 4.2). The resulting graph H is an (ℓ,d)-graph by Lemma 4.1. Since $d(H)=d$, H is an ℓ-diameter stable graph. Note that H is not an ℓ-distance stable graph.
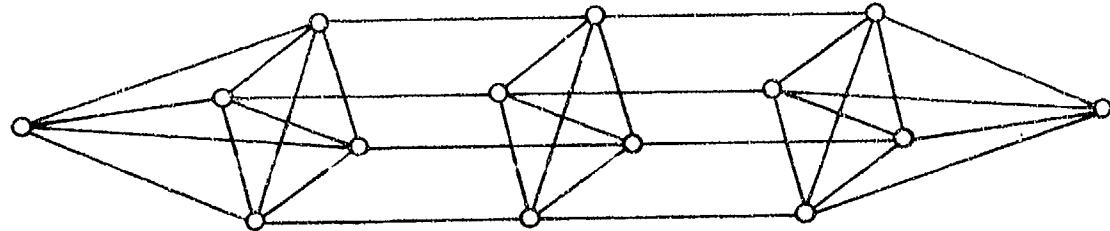

Figure 4.2. A 4-Diameter Stable Graph With Diameter 4.

<u>Example 4.2.4.</u>

In [17], $G_E(n,\delta,d,\ell-1)$ was defined as the class of (ℓ,d)-graphs with diameter $\leq \delta$ on n vertices, $g(n,\delta,d,\ell-1)$ denoted the minimal number of edges of graphs within $G_E(n,\delta,d,\ell-1)$.

U. S. R. Murty proved [16] that if

$$n > \frac{(3+\sqrt{5})\ell}{2} \text{ , then}$$

$$g(n,2,2,\ell-1) = \ell\left(n - \frac{\ell+1}{2}\right), \tag{4.1}$$

and the corresponding unique extremal graph is obtained from the complete bipartite graph $K_{\ell,n-\ell}$ by adding all the edges within the class of ℓ vertices (Fig. 4.3). The resulting graph is denoted as $\Gamma_n(\ell)$.
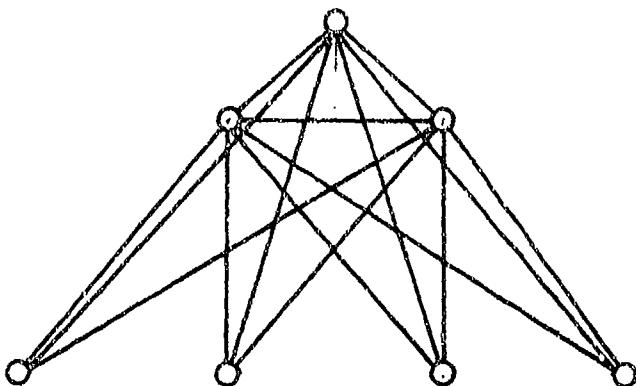
Figure 4.3. A 3-Diameter Stable Graph With Diameter 2.

Using the previous construction ℓ-diameter stable graphs with even diameter and arbitrary number of vertices can be obtained by taking $\left[\frac{d}{2}\right]$ disjoint copies of $\Gamma_n(\ell)$ and in each two successive $\Gamma_n(\ell)$'s identify two vertices from the class of n-ℓ vertices, (Fig. 4.4). The resulting graph is by Lemma 4.2 ℓ-diameter stable with diameter $2\left[\frac{d}{2}\right]$.

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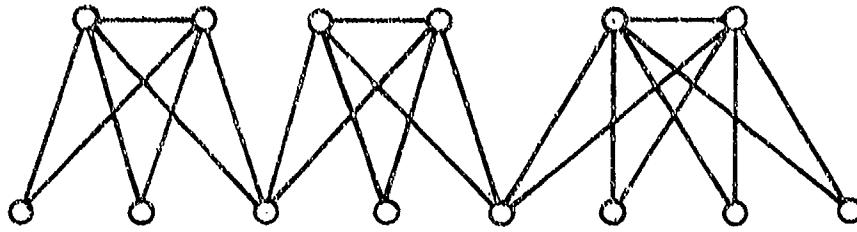
Figure 4.4. A 2-Diameter Stable Graph with Diameter 6.

Using this construction an upper bound on $g(n,2m,2m,\ell-1)$ is obtained, by using (4.1), and the fact that an ℓ-diameter stable graph is in particular ℓ-edge connected, yields a lower bound. Thus,

$$\frac{n\ell}{2} \leq g(n,2m,2m,\ell-1) \leq \ell\left(n-1 - \frac{m(\ell-1)}{2}\right).$$

Since,

$$g(n,2m+1,2m+1,\ell-1) \leq g(n,2m,2m,\ell-1),$$

we have,

$$\frac{n\ell}{2} \leq g(n,d,d,\ell-1) \leq \ell\left(n-1 - \left[\frac{d}{2}\right](\ell-1)\frac{1}{2}\right). \tag{4.2}$$

Asymptotically, applying the tightness measure to (4.2) we obtain

$$\frac{\ell}{2} \leq \lim_{n\to\infty} \frac{g(n,d,d,\ell-1)}{n} \leq \ell. \tag{4.3}$$

The previous graph is not ℓ-distance stable.

Example 4.2.5.

ℓ-distance stable graphs with diameter d are obtained as follows. Take d-1 disjoint copies of the complement of K_ℓ H_1, H_2, $\ldots$, H_{d-1}, where the vertices of H_j are denoted by v_{1_j}, v_{2_j}, $\ldots$, v_{ℓ_j}, $\left(1 \leq j \leq d-1\right)$. Join v_{m_j} to $v_{n_{j+1}}$ for all $1 \leq m$, $n \leq \ell$ and $1 \leq j \leq d-1$.

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Then join new vertices u_1 and u_2 adjacent to all vertices of H_1 and H_{d-1} respectively. The resulting graph has diameter d and by Lemma 4.3 is also an ℓ-distance stable graph, (Fig. 4.5).
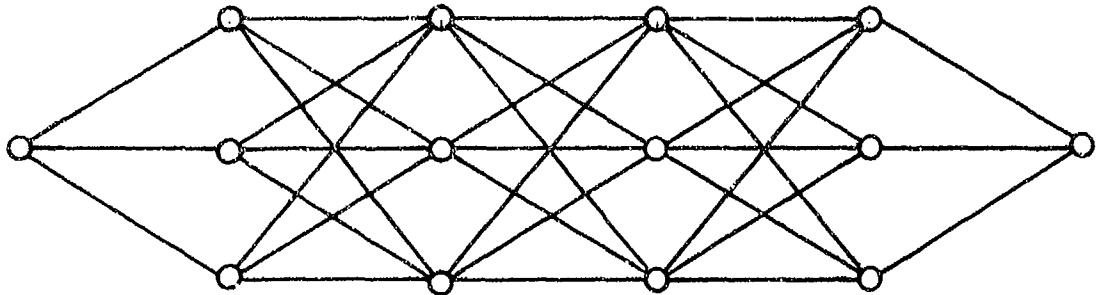
Figure 4.5. A 3-Distance Stable Graph of Diameter 5.

By adding new vertices all adjacent to the vertices of H_{d-1}, ℓ-distance stable graphs with diameter d having prescribed number of vertices may be constructed.

As can be seen from these examples and many others, there are "many" different (ℓ,d)-graphs and ℓ-diameter stable graphs. In the next section some of their properties will be investigated.

4.3 Some Results for (ℓ,d)-graphs

An edge $e \in E(G)$, will be called <u>cyclic</u> if there exists a cycle in G containing e. To each cyclic edge assign a natural number $g(e) \geq 3$, which is the length of the shortest cycle in G containing e. If e is a bridge then $g(e) \underset{=}{\triangle} \infty$. The <u>girth</u> of G, girth (G), is defined as

$$\text{girth}(G) \triangleq \underset{e \in E(G)}{\text{Min}} \quad g(e).$$

The following properties of (ℓ,d)-graphs are quite obvious.

<u>Theorem 4.4.</u> Let G be an (ℓ,d)-graph, $(\ell,d \geq 2)$, then

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$$g(e) \le d+1, \quad \forall e \in E(G) \qquad\qquad (4.4)$$

and this result is best possible.

   Proof. If G is an (ℓ,d)-graph ($\ell \ge 2$), then G is in particular 2-edge connected and hence all of its edges are cyclic. Let $ab = e$ be an edge of G and assume

$$g(e) > d+1. \quad \text{Then } d(G-e) \ge d_{G-e}(a,b) > d.$$

   To show that (4.4) best possible, consider the cycle C_{d+1} which is a $(2,d)$-graph and $g(e) = d+1 \quad \forall e \in E(C_{d+1})$.

<div align="right">Q.E.D.</div>

   Theorem 4.5. Let G be an ℓ-diameter stable graph, $\ell \ge 2$, then

$$g(e) \le d(G) + 1, \quad \forall e \in E(G). \qquad\qquad (4.5)$$

Furthermore, (4.5) is best possible.

   Proof. G is 2-edge connected and if $g(e) > d(G) + 1$ then

$$d(G-e) \ge d_{G-e}(a,b) > d(G),$$

where
$$e = ab,$$

contradicting the assumption that G is an ℓ-distance stable graph.

   To show that (4.5) cannot in general be improved we construct for any arbitrary integer $d \ge 2$ an ℓ-diameter stable graph with at least one edge e such that $g(e) = d(G) + 1$. Let G_1 and G_2 be distinct ℓ-diameter stable graphs with diameters d_1 and d_2 respectively, and let x_i, y_i be a diametrical pair of vertices of G_i ($i=1,2$). Assume in addition that G_i contains a triple x_i, y_i, z_i such that

$$d_i = d_{G_i}(x_i, y_i) = d_{G_i}(x_i, z_i) = d_{G_i}(y_i, z_i), \quad i=1,2.$$

<div align="center">58</div>

By Lemma 4.2, the graph G' generated from G_1 and G_2 by identifying the vertex $x_1 \in V(G_1)$ with the vertex $x_2 \in V(G_2)$ is an ℓ-diameter stable graph with diameter $d_1 + d_2$. Finally define the graph $G = G + y_1 y_2$, obtained from G by joining the vertices y_1 and y_2 by an edge $y_1 y_2$. G is clearly an ℓ-distance stable graph which contains an edge $y_1 y_2$ such that $g(y_1 y_2) = d(G) + 1$, where $d(G) = d_1 + d_2$ by the above requirement of three diametrical pairs in each graph.

Q.E.D.

Inequality (4.5) (and also 4.4) does not yield a sufficient condition for a graph G to be an ℓ-diameter stable graph. Take for instance the graph H, composed of 3 5-cycles C_1, C_2, C_3 with vertices v_i^k $1 \leq i \leq 5$, $1 \leq k \leq 3$, such that

$$C_1 \cap C_2 = v_2' v_3' , \quad C_1 \cap C_3 = v_4' \, v_5' \text{ and } C_2 \cap C_2 = \phi.$$

Although every edge of H is contained in a cycle of length $\leq d(H)+1$, $d(H - v_3' \, v_4') = 6 > d(H) = 5$ (Figure 4.6).
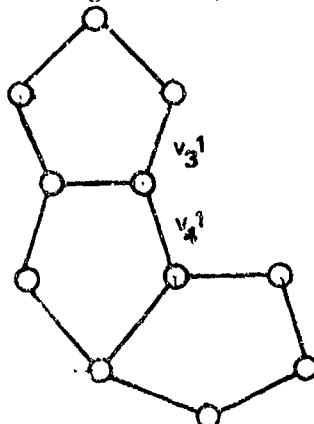
Figure 4.6. A Counter Example.

In particular we conclude from (4.5) that if G is ℓ-diameter, stable, then,

$$\text{girth}(G) \leq d(G) + 1. \tag{4.6}$$

We do not know whether for d > 4, (4.6) can be improved. In other words if one can construct for an arbitrary natural number d an ℓ-diameter stable (or 2-diameter stable) graph G such that d(G) = d and g(e) = d(G) + 1 ∀ e ∈ E(G). If d=2 (4.6) cannot be improved. If d=3, 4 Figures 4.7 and 4.8 respectively show a realization for (4.6).
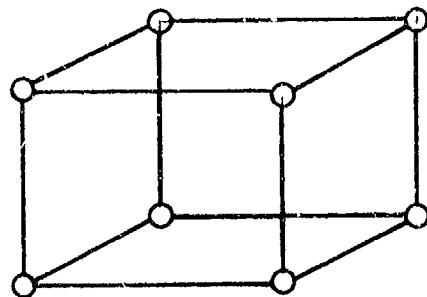
Figure 4.7. A 2-Diameter Stable Graph
with Diameter 3 and Girth 4.
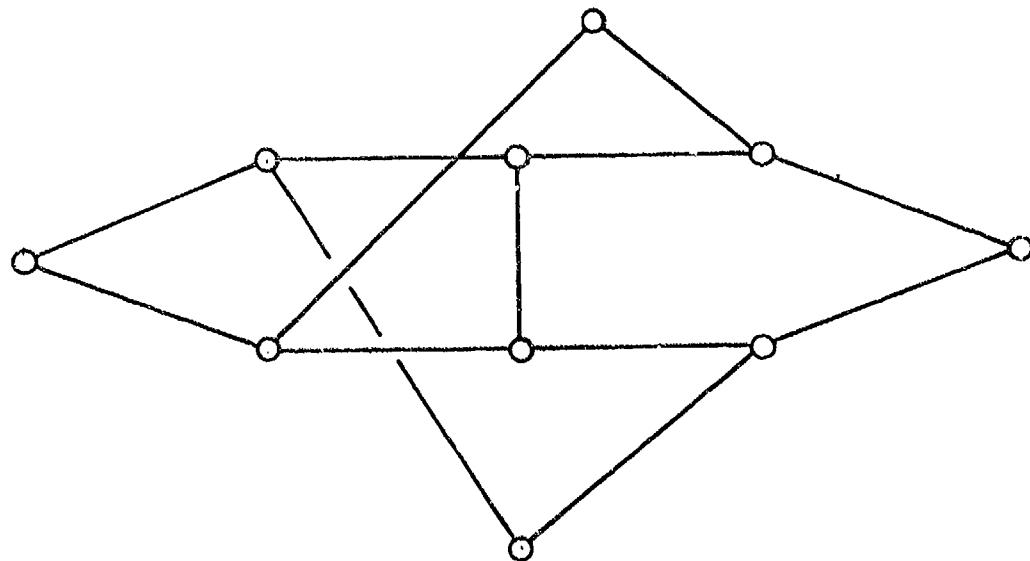
Figure 4.8. A 2-Diameter Stable Graph with Diameter 4 and Girth 5.

The next theorem gives a NASC for a pair of vertices to be an ℓ-distance pair.

Theorem 4.6. A pair of nonadjacent vertices $x, y \in V(G)$ is an ℓ-distance pair if and only if there are at least ℓ edge disjoint x, y-paths of length $d_G(x, y)$ in G.

**Proof.** Clearly, if there are at least ℓ edge disjoint x, y-paths of length $d_G(x, y)$ in G, then x, y is an ℓ-distance pair. To prove that if x, y is an ℓ-distance pair then there are at least ℓ edge disjoint x, y-paths of length $d_G(x, y)$ in G, we use the digraph version of Menger's Theorem (see [9] Chap. 5). The only part of the graph G relevant to the proof of the theorem is the set W of the x, y-paths of length $d_G(x, y)$. We can direct the edges of W to form a directed graph D on W with the property that the directed x, y-paths correspond exactly to the x, y-paths in W. By applying the digraph version of Menger's theorem with respect to edges one obtains that the maximum number of edge disjoint directed paths of length $d_G(x, y)$ from x to y is equal to the minimum number of edges whose removal from D cuts all directed x, y-paths in W. This is exactly an equivalent statement to the theorem. The vertex version of this theorem was done in [23].

A constructive proof of the theorem can also be given. First consider the case $\ell=2$.

Define the following sets.

$$A_i(x) \triangleq \left\{ y \in V(G): \quad d_G(x, y) = i \right\} \quad 0 \le i \le d(G),$$

and

$$E_i(x) \triangleq \left\{ e \in E(G): \quad ab=e, \ a \in A_i(x), \ b \in A_{i+1}(x), \right\} \quad 0 \le i \le d(G) - 1.$$

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Since x,y is a 2-distance pair there are obviously at least two x,y-paths of length $d_G(x,y)$ in G. We shall prove that one can always find at least two _edge disjoint_ such paths. If $d_G(x,y) = 2,3$ the proof follows immediately. Assume therefore $d_G(x,y) > 3$, and let $d_G(x,y) = d_o$. If there are no two edge disjoint x,y-paths of length d_o in G, there must be a maximal integer i ($1 < i < d_o$) so that there are two x,y-paths P_1 and P_2 of length d_o with edge disjoint subpaths from x to a vertex x_i, in $A_i(x)$, (note that P_1 and P_2 may have common edges in $E_m(x)$ only if $m \geq i$). If $i=0$ x,y is not a 2-distance pair, and $i=1$ is impossible in a graph without multiple edges. Let $e \in E_i(x)$ be an edge contained both in P_1 and P_2, with end vertices x_i and x_{i+1}, ($x_i \in A_i(x)$, $x_{i+1} \in A_{i+1}(x)$). Since x,y is a 2-distance pair, there must be a vertex $y_i \in A_i'(x)$ and a y_i,y-path of length d_o-i not containing e. Otherwise, $d_G(x,y) < d_{G-e}(x,y)$. Denote this path by P_3'. By definition there exists an x,y_i-path P_4' of length i. If $P_1(P_2)$ and P_4' are edge disjoint, define $P = P_3' \cup P_4'$. $P_1(P_2)$ and P are two x,y-paths of length d_o, such that if there exists $e_1 \in E_m(x)$ and $e_1 \in P_1 \cap P$ ($e_1 \in P_2 \cap P$) then $m > i$. This contradicts the maximality of i.

If P_4', P_1 and P_4', P_2 have common edges let $k < i$ be the greatest integer such that there is an edge $e_2 \in E_k(x)$ and $e_2 \in P_1 \cap P_4'$ or $e_2 \in P_2 \cap P_4'$, $e_2 = x_k x_{k+1}$, $x_k \in A_k(x)$ and $x_{k+1} \in A_{k+1}(x)$. Define a new path P' composed of P_1 resp. P_2 from x to x_k and of P_4' from x_k to y_i. As before the paths P_2 resp. P_1 and P' contradict the maximality of i. This completes the proof for $\ell=2$. The same type of constructive proof for $\ell > 2$ is valid but seems to be too tedious to present here.

Q.E.D.

Since any diametrical pair of vertices in an ℓ-diameter stable graph must be an ℓ-distance pair, the following necessary condition for ℓ-diameter stable graphs is obtained by applying Theorem 4.6.

Theorem 4.7. If G is an ℓ-diameter stable graph and x,y is a diametrical pair of vertices in G, then there are at least ℓ edge disjoint x,y-paths of length d(G) in G.

Note that the statement in Theorem 4.7 cannot serve as a sufficient condition for ℓ-diameter stability, as can be seen for instance from a cycle of even length, C_{2m}, which is not 2-diameter stable, but each diametrical pair of vertices in C_{2m} is joined by two edge disjoint paths of length m, which is the diameter of C_{2m}.

We conclude this section by stating a necessary and sufficient condition for a graph to be ℓ-distance stable, which follows by Lemma 4.3 and Theorem 4.6.

Theorem 4.8. A graph G is ℓ-distance stable if and only if between any pair of nonadjacent vertices $x,y \in V(G)$ there are at least ℓ edge disjoint paths of length $d_G(x,y)$.

There is a clear analogy between Theorem 4.8 characterizing ℓ-distance stable graphs and the edge version of the Menger-Whitney Theorem (see [9], Chapter 5), derived by Ford and Fulkerson, Elias, Feinstein, Shanon, Kotzig and others.

It should be noted that due to the similarity between ℓ-distance stable and ℓ edge connected graphs, other analogs of connectivity theorems may be proved for ℓ-distance stable graphs (e.g., the Dirac's fan theorem).

The next section is devoted to (2,d)-graphs.

4.4 Critical and Superfluous Edges and 2-Diameter Stable Graphs

An edge $e \in E(G)$ is called __superfluous__ if $d(G-e) = d(G)$, otherwise e is said to be a (diameter) __critical__ edge. By definition, a graph is 2-diameter stable if a   only if all its edges are superfluous. A graph is called (diameter) critical if all its edges are critical. Critical graphs were studies in [29] − [32].

Examples of critical edges are a bridge, any edge of C_n and any edge of K_n. On the other hand all edges of the diameter stable graphs mentioned in Section 4.2 are of course superfluous. A few properties of superfluous and critical edges and consequently results on $(2,d)$-graphs are now given.

__Theorem 4.9.__ If $e \in E(G)$ is a superfluous edge of a graph G, then

$$g(e) \leq d(G) + 1 \tag{4.7}$$

and this result is best possible.

__Proof.__ If e is superfluous it is not a bridge and therefore it is a cyclic edge. Let $e = ab$ and assume $g(e) > d(G) + 1$. But then $d(G-e) \geq d_{G-e}(a,b) > d(G)$ contradicting the assumption that e is superfluous. To show that (4.7) is best possible, take a cycle of length $d+1$ ($d \geq 3$, integer) and a path of length $\left\lceil \dfrac{d}{2} \right\rceil$ sharing exactly one vertex with the cycle (Figure 4.9).

Figure 4.9. Superflows Edges.

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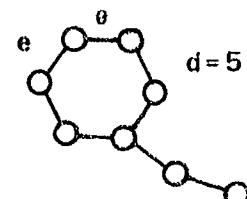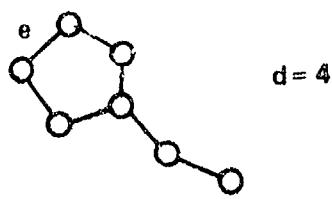
The resulting graph is of diameter d and contains at least one super-fluous edge whose minimal cycle is of length d + 1.

It is easy to see that the converse of Theorem 4.9 is not true. The edges denoted by e' in Figure 4.10 are critical although their minimal cycle is of length less than or equal to d(G) + 1.
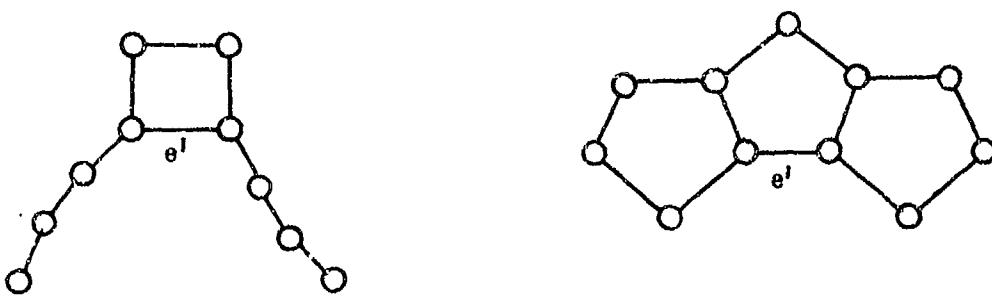
Figure 4.10.   Critical Edges.

A simple consequence of Theorem 4.9 is Theorem 4.10.

Theorem 4.10.   If for an edge $e \in E(G)$ of a graph G

$$g(e) > d(G) + 1, \tag{4.8}$$

then e is critical.

Theorem 4.11.   If $e \in E(G)$ is a cyclic edge of a graph G, then

$$d(G) \leq d(G-e) \leq 2d(G). \tag{4.9}$$

Proof.   The left hand side of (4.9) is obvious.

Let

$$x, y \in V(G)$$

be a diametrical pair of vertices in G-e, and define

$$A_i = \left\{ v \in V(G): \quad d_{G-e}(x,v) = i \right\} \quad 0 \leq i \leq d(G-e).$$

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Let $e = ab$, and let $a \in A_m$, $b \in A_n$ $0 \leq m \leq n \leq d(G-e)$. Define $\ell = \left[\dfrac{n-m}{2}\right]$ and take a vertex $z \in A_{m+\ell}$. There exists an x,z-path P_1 in G-e of length $m+\ell$, and a z,y-path P_2 in G-e, of length $d(G-e) - m-\ell$. The shortest x,z-path in G does not contain e, otherwise it would be of length at least $m+1+\ell$. Similarly the shortest z,y-path in G does not contain e, since otherwise it will be of length at least $d(G-e)-n+1+\ell$, which is not shorter than $d(G-e)-m-\ell$ due to the specific value we have chosen for ℓ. Therefore the length of P_1 and P_2 does not exceed $d(G)$ and $P_1 \cup P_2$ yields an x,y-path of length not exceeding $2d(G)$ in the graph G-e, which proves (4.9).

To show that (4.9) cannot be improved take an odd cycle C_{2k+1}.

$$2k = d(C_{2k+1} - e) = 2d(C_{2k+1}) = 2k.$$

<div align="right">Q.E.D.</div>

If G is 2-edge connected, then

$$d(G) \leq d(G-e) \leq 2d(G), \qquad \forall e \in E(G).$$

We thus conclude the following,

Theorem 4.12. All 2-edge connected graphs G are $(2,2d(G))$-stable.
Another simple bound on $d(G-e)$, where e is a cyclic edge is,

$$d(G-e) \leq d(G) + g(e) - 2.$$

We thus obtain the following property.

Theorem 4.13. If G is a 2-edge connected graph and $g(e) \leq g$ $\forall e \in E(G)$ for some integer g, then,

$$d(G-e) \leq d(G) + g-2 \qquad \forall e \in E(G).$$

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The following theorem gives a sufficient condition for a graph to be a (2,d)-graph.

Theorem 4.14. If for some integer $m \geq 0$, we have for a graph G,

(1) $g(e) \leq 3+m$, $\forall e \in E(G)$.

(2) Any pair of vertices $x,y \in V(G)$, such that $d_G(x,y) \geq d \cdot m$

is joined by at least two edge disjoint paths of length

not exceeding d;

Then G is a (2,d)-graph.

Proof. Take any $e \in E(G)$ and let $x,y \in V(G-e)$. If $d_G(x,y) \geq d-m$, then by (2) $d_{G-e}(x,y) \leq d$. Assume therefore, $d_G(x,y) \leq d-m-1$, then by (1) $d_{G-e}(x,y) \leq 3+m-1+d-m-2=d$.

Q.E.D.

Theorem 4.15 is a consequence of Theorem 4.14 for m=0 and Theorem 4.7.

Theorem 4.15. Let G be a graph such that $g(e) = 3$ $\forall e \in E(G)$. Then G is 2-diameter stable if and only if every pair of diametrical vertices in G is joined by at least two edge disjoint paths of length $d(G)$.

By Theorem 4.15, the following class of graphs with arbitrary diameter, is a class of 2-diameter stable graphs. Take a cycle with even number of vertices labeled $v_1, v_2, \ldots, v_{2m}$, to each pair of adjacent vertices v_1, $v_{i+1 \pmod{2n}}$, $1 \leq i \leq 2n$ join a vertex w_i, such that w_i is adjacent to v_1 and to $v_{i+1 \pmod{2n}}$, $1 \leq i \leq 2n$. The resulting graph H_{2n} (Fig. 4.11) is of diameter n+1, and is by Theorem 4.15 2-diameter stable graph.

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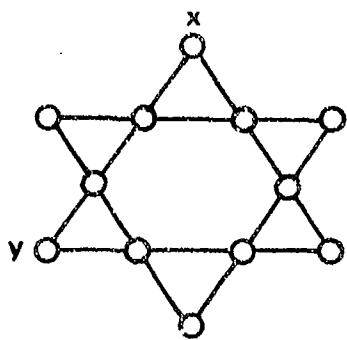
Figure 4.11. H_6.

The graph H_{2n} shows that Lemma 4.1 is not a necessary condition for 2-diameter stability, since for instance the vertices x and y (Fig. 4.11) are <u>not</u> joined by two edge disjoint paths of length $\leq n+1 = d(H_{2n})$. By joining in H_{2n}, w_1 to w_2, w_3 to w_4 etc., one obtains a graph H_{2n} with diameter n+1 which is 3 diameter stable and does not satisfy the converse of Lemma 4.1 (Fig. 4.12).
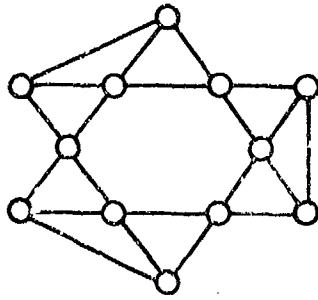

Figure 4.12. $\overline{H}_6$.

Fig. 4.11 shows a 2-diameter stable graph of diameter 4 which does not satisfy the converse of Lemma 4.1. If d < 4 then Lemma 4.1 is a necessary and sufficient condition for 2-diameter stability, as stated in the next theorem.

Theorem 4.16. A graph G with diameter 2 or 3 is 2-diameter stable if and only if there are at least two edge disjoint paths of length not exceeding d(G), between any pair of vertices of G.

Proof. Sufficiency follows from Lemma 4.1. If $d(G) = 2$ then the necessity follows from Theorems 4.4 and 4.7. If $d(G) = 3$ then for any pair of vertices $x, y \in V(G)$ such that $d_G(x,y) = 1,3$, the necessity again is derived from Theorems 4.4 and 4.7. Therefore, let $x, y \in V(G)$ be two vertices such that $d_G(x,y) = 2$, and denote the shortest x,y-path by P_1 ($P_1 = xzy$). Since G is 2-diameter stable, there exists an x,y-path $P_2 \neq P_1$ and $2 \leq |P_2| \leq 3$. If $|P_2| = 2$ then P_1 and P_2 are edge disjoint. If all x,y-paths ($\neq P_1$) are of length 3 and none of them is disjoint from P_1, take such a path $P_3 = xaby$, and assume without loss of generality that $xz = xa$. There must be an x,y-path P_4 of length 3 ($P_4 = x c dy$) such that $xc \neq xz$, otherwise $d_{G-xc}(x,y) > 3$, contradicting the 2-diameter stability of G. In this case P_3 and P_4 are edge disjoint x,y-paths of length 3.

Q.E.D.

4.5   Some Extremal Problems for 2-Diameter Stable Graphs

Similar to the classes defined in Chapter 2, we denote by $H_s(k,d)$ the class of all k-diameter stable graphs with diameter d, and by $H_s(n,k,d)$ the subclass of $H_s(k,d)$ containing all graphs with exactly n vertices.

Let

$$f_s(k,d) \triangleq \underset{G \in H_s(k,d)}{\text{Min}} |V(G)|,$$

$$g_s(k,d) \triangleq \underset{G \in H_s(k,d)}{\text{Min}} |E(G)|,$$

and

$$g_s(n,k,d) \underset{=}{\Delta} \underset{G \in H_s(n,k,d)}{\text{Min}} |E(G)| .$$

As before we call the graphs in $H_s(k,d)$, $H_s(k,d)$-graphs etc., and the $H_s(k,d)$-graphs with minimal number of vertices and edges are called vertex extremal and extremal graphs of the respective classes. As mentioned before Murty [16] showed that if $n > \frac{(3+\sqrt{5})k}{2}$, then

$$g_2(n,k,2) = k\left(n - \frac{k+1}{2}\right) ,$$

and obtained the unique $H_s(n,k,2)$-extremal graph realization (see Figure 4.3).

In this section we deal only with the case k=2. First the value of $f_s(2,d)$ is given.

Since there is no 2-diameter stable with diameter = 1, we set $d \geq 2$.

<u>Theorem 4.17.</u> For $d \geq 2$

$$f_s(2,d) = \left[\frac{d+1}{2}\right] + d + 1. \tag{4.10}$$

<u>Proof.</u> Let $G \in H_s(2.d)$ have a diametrical arc x_0, $x_1,\ldots,$ x_d and

$$V_i \underset{=}{\Delta} \left\{ x \in V(G) : d(x_0,x) = i \right\} \quad , \quad 0 \leq i \leq d.$$

Define

$$n_i = |V_i| , \qquad 0 \leq i \leq d.$$

G is bridgeless and hence $n_i + n_{i+1} \geq 3 \qquad \forall \ 0 \leq i \leq d-1.$

Therefore

$$\sum_{i=0}^{d} n_i \geq \left[\frac{d+1}{2}\right] + d + 1$$

and,

$$f_s(2,d) \geq \left\lceil \frac{d+1}{2} \right\rceil + d + 1.$$

On the other hand consid.r the following class of graphs of arbitrary diameter $d \geq 2$, as in Fig. 4.13, which by Lemma 4.1 are 2-diameter stable.
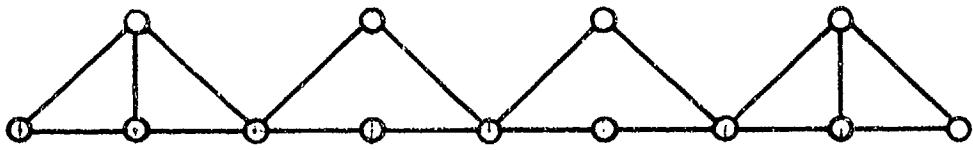
An $H_S(2,d)$ — Graph with Even Diameter (8).
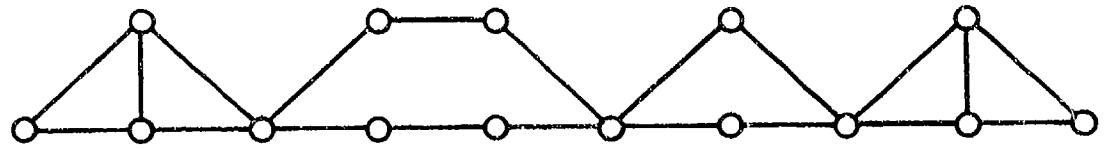
An $H_S(2,d)$ — Graph with Odd Diameter (9).

Figure 4.13.

This class shows

$$f_s(2,d) \leq \left\lceil \frac{d+1}{2} \right\rceil + d + 1.$$

and (4.10) follows.

Q.E.D.

Theorem 4.18. $g_s(2,2) = 5$, and for $d \geq 3$

$$g_s(2,d) = 2(d+1) \tag{4.11}$$

Proof. The value of $g_s(2,2)$ can easily be verified. The graphs described in Fig. 4.13 show

$$g_s(2,d) \leq 2(d+1). \tag{4.12}$$

On the other hand if

$$G_i = \left\{ (u,v) \in E(G) : u \in V_i, v \in V_{i+1} \right\} \quad 0 \leq i \leq d-1,$$

then $|G_i| \geq 2 \quad \forall \, 0 \leq i \leq d-1$, otherwise elimination of a single edge disconnects the graph.

Therefore, $2d \leq g_s(2,d)$.

Let $\{x_0\} = V_0 \quad \{x_1, y_1\} = V_1 \quad (x_1 \neq y_1), \quad x_2, y_2 \in V_2$

assume $x_0 y_1, \; x_0 x_1 \in G_0, \quad x_1 x_2, \; y_1 y_2 \in G_1$, where V_i are defined for a 2-diameter stable graph G, as in the proof of Theorem 4.17, (G is a vertex extremal $H_s(2,d)$-graph). If $x_1 y_1 \notin E(G)$ then there must be an edge in G_1 ($\neq x_1 x_2$) incident with x_1, otherwise $d_{G-x_1 x_2}(x_1, x_d) > d \implies |G_1| \geq 3$. If $x_1 y_1 \in E(G)$ then again there is an extra edge in G. The same argument applies to the G_{d-2}, G_{d-1} and therefore $2(d+1) \leq g_s(2,d)$, which together with (4.12) proves (4.11).

Q.E.D.

Theorem 4... gives asymptotic bounds on $g_s(n,2,d)$.

Theorem 4.19. For $d \geq 4$ and $n \geq \left\lceil \dfrac{d+1}{2} \right\rceil + d + 1$,

$$1 \leq \lim_{n \to \infty} \frac{g_s(n,2,d)}{n} \leq \begin{cases} \dfrac{d}{d-2} & d\text{-odd} \\[2mm] \dfrac{d-1}{d-3} & d\text{-even.} \end{cases} \tag{4.13}$$

Proof. The lower limit in (4.13) is simply due to the 2-edge conn. of any $H_s(n,2,d)$-graph. To show the upper limit we start with the $H_s(2,d)$-extremal graphs shown in Fig. 4.13 and add to them vertices so that the resulting graph is a 2-diameter stable with diameter d. Denote by $x_0, x_1, \ldots, x_d$ the vertices of a diametrical arc of the $H_s(2,d)$-extremal graph described previously.

Let d be even, then consider two cases.

Case 1:   d=4m   ,   m $\geq$ 1.

Connect x_{2m-1}, x_{2m+1} by edge disjoint paths of length 2m, and possibly one single path of length less than 2m, so that the resulting graph has exactly n vertices (Fig. 4.14).
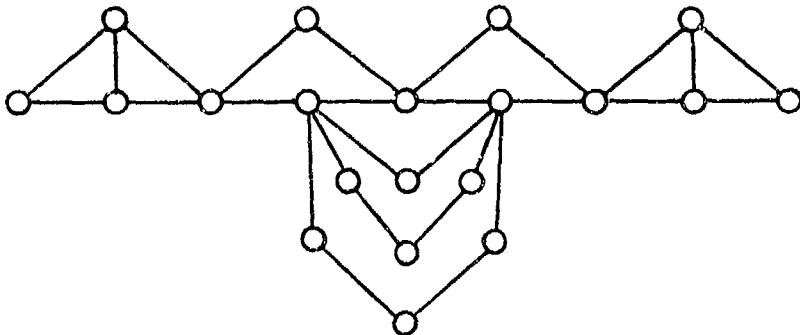

Figure 4.14.   An H_S (20,2,8) — Graph.

The graph obtained is clearly 2-diameter stable with diameter d with

$8m+2+ \left[\dfrac{n-6m-1}{2m-1} \right]$ 2m+t+1, edges, where n-6m-1 $=$ (2m-1) s + t, $0 \leq t \leq 2m$

Hence,

$$g_s(n,2,4m) \leq 8m + \left[\dfrac{n-6m-1}{2m-1} \right] 2m + t + 3, \qquad (4.14)$$

where t < 2m-1.

Case 2:   d $=$ 4m-2   ,   m $\geq$ 1.

Similar to Case 1 we obtain a family of $H_s(n,2,4m+2)$-graph of diameter 4m+2, (Fig. 4.15), with $8m+6 + \left[\dfrac{n-6m-4}{2m} \right]$ (2m+1) + t + 1, where n-6m-4 $=$ 2m s + t, $0 \leq t < 2m$.
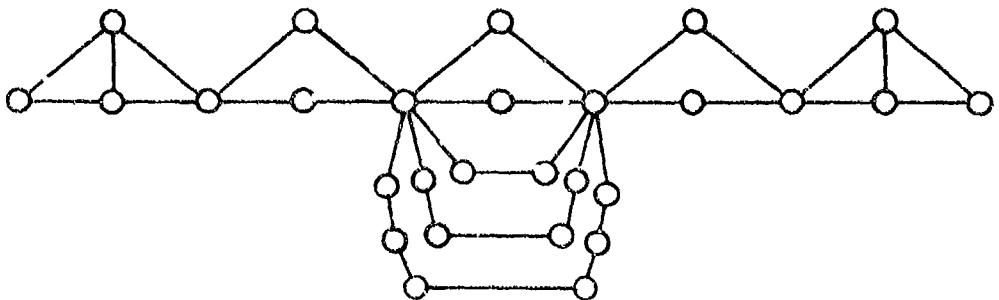
Figure 4.15.   An H_S (28,2,10) — Graph.

Hence,

$$g_s(n,2,4m+2) \leq 8m + \left[\frac{n-6m-4}{2m}\right](2m+1) + t + 7 \qquad (4.15)$$

where $t < 2m$.

If in inequalities (4.14) and (4.15) we substitute the appropriate values of m, we obtain in both cases after dividing both sides by n and taking the limit as $n \to \infty$

$$\lim_{n \to \infty} \frac{g_s(n,2,d)}{n} \leq \frac{d}{d-2} \ ,$$

which proves the upper bound for (4.13) when d is even.

If d is odd the following construction (Fig. 4.16) yields an $H_2(n,2,d)$-graph.
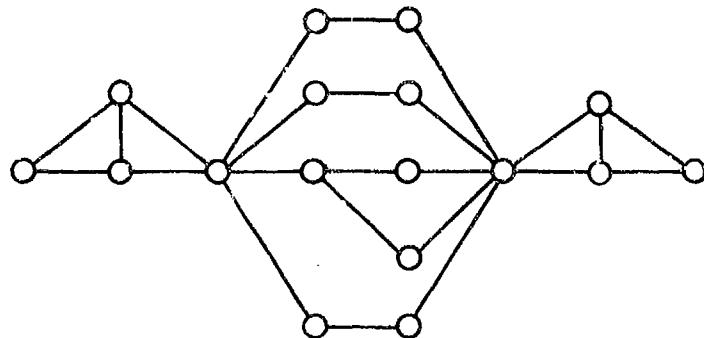
Figure 4.16.   An H_S (17,2,7) — Graph.

By counting the number of edges of the resulting graph, one obtains the upper bound in (4.13) for d odd.

Q.E.D.

CONCLUSIONS AND FURTHER PROBLEMS

In this dissertation, we have considered graph theoretical problems motivated by questions related to the design of reliable communication networks with a bounded maximal message delay.

In the first part of the research, consisting of Chapter 2 and Chapter 3, extremal problems related to the classes of graphs $H_i(k,d)$ and $H_i(n,k,d)$ ($i = 1,2,3$), defined in Sections 2.1 and 3.1, were studied. For the related functions we have obtained the following results.

(1) $f_i(k, \ell+1) = (k+1)(\ell+1) + i$, $i = 0,1,2$, $k \geq 2$, $\ell \geq 1$.

(2) $\dfrac{f_1(k,d)k}{2} \leq g_1(k,d) \leq \dfrac{f_1(k,d)k}{2} + 1$.

In some cases the lower bound in (2) was shown to be attained. It seems that the lower bound may be attained in other cases as well, but we were not able to prove that.

(3) For $k, d \geq 2$,

$$\underset{k,d \geq 2}{\text{Max}} \left\{ f_1(k,d), \ 2(k+1)+(d-3)\sqrt{k} \right\} \leq f_2(k,d) \leq \begin{cases} \dfrac{(k+1)d}{2} + 1 & , \ d\text{-even} \\[2mm] \dfrac{(k+1)(d+1)}{2} & , \ d\text{-odd.} \end{cases}$$

(4)

$$kd \leq g_2(k,d) \leq \begin{cases} \dfrac{dk(k+2)}{4} & , \ d\text{-even, } k\text{-even} \\[2mm] \dfrac{k(k(d+1)+2(d-1))}{4} & , \ d\text{-odd, } k\text{-even} \\[2mm] \dfrac{(k+1)(4k+(d-3)(k+1))}{4} & , \ d\text{-odd, } k\text{-odd} \\[2mm] \dfrac{d(k+1)^2}{4} & , \ d\text{-even, } k\text{-odd .} \end{cases}$$

In special cases tighter bounds on $f_2(k,d)$ and $g_2(k,d)$ can be obtained.

(5) $f_3(k,d) = k(d-1) + 2$, k, $d \geq 1$.

(6) $g_3(k,d) = \left[\dfrac{(k(d-1)+2)k+1}{2} \right]$ $\quad\quad$ $k, d \geq 2$.

(7) For $i = 1, 2, 3,$ $\quad$ $(d > 1)$

$$ n \leq g_i(n, 2, d) \leq n + \left[\frac{n-d-2}{d-1} \right] , $$

and asymptotically,

$$ 1 \leq \lim_{n \to \infty} \frac{g_i(n, 2, d)}{n} \leq 1 + \frac{1}{d-1} . $$

(8) For $k = 3$, and $i = 1, 2, 3,$

$$ \frac{3}{2} \leq \lim_{n \to \infty} \frac{g_i(n, 3, d)}{n} \leq \frac{3}{2} + \frac{1}{2} \frac{1}{2^{\left[\frac{d}{2}\right]} - 1} . $$

(9) For $k > 3$, $d \geq 2$, $n \geq (v_{\left[\frac{d}{2}\right], k} - 1)k + 1$,

$$ \frac{k}{2} \leq \lim_{n \to \infty} \frac{g_1(n, k, d)}{n} \leq \frac{k}{2} + \frac{1}{2} \cdot \frac{k-2}{(k-1)^{\left[\frac{d}{2}\right]} - 1} . $$

For $i = 2, 3,$

$$ \frac{k}{2} \leq \lim_{n \to \infty} \frac{\overline{g}_i(n, k, d)}{n} \leq \frac{k}{2} + \frac{1}{2} \frac{k-2}{(k-1)^{\left[\frac{d}{2}\right]} - 1} . $$

(10) For

d, $\ell \geq 2$ and $(d-1)2\ell + 2 \leq n \leq (d-1)(\ell-1)4 + 2$, we have shown that

$$ g_1(n, 2\ell, d) = n\ell . $$

Considering network design applications, the previously listed results may be used to calculate the "additional cost", in terms of vertices and edges, required for a higher reliability constraint measure on the network. Thus, for instance, if a network with diameter d and minimal degree k is to be "modified" to a k-vertex connected network, at most $(\ell-1)(2k-1) + i(k+1)$ $(d=3\ell+i, i=0,1,2)$ vertices must be added to the network. Note that this modification may require a new arrangement of the vertices and edges.

The obvious further problems to be investigated as suggested by the latter results are, of course, those of finding the exact values of the respective function, for which only bounds are given here.

In the second part of this research, Chapter 4, new reliability criteria, motivated by maximal message delay considerations in communication networks, were defined and analyzed. (ℓ,d)-graphs, ℓ-diameter stable and ℓ-distance stable graphs were introduced (see Section 4.1) and different classes of those graphs were constructed. The main results concerning those classes are listed below.

I.   Concerning (ℓ,d)-graphs we obtained;

(1)   For an (ℓ,d)-graph G

$g(e) \leq d + 1, \forall e \in E(G),$

where $g(e)$ was defined as the minimal cycle containing an edge $e \in E(G)$.

(2)   All 2-edge connected graphs G were proven to be $(2, 2d(G))$-graphs.

(3)   The following, is a sufficient condition for a graph to be a $(2,d)$-graph;

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If for some integer $m \geq 0$, we have for a graph G,

(a) $g(e) \leq 3 + m$, $\quad \forall e \in E(G)$.

(b) Any pair of vertices x, y $\in$ V(G), such that $d_G(x,y) \geq d-m$ is joined by at least two edge disjoint paths of length not exceeding d.

Then G is a (2,d)-graph.

II. For ℓ-diameter stable graphs we obtained the following:

(1) If G is an ℓ-diameter stable graph, $\ell \geq 2$, then

$$g(e) \leq d(G) + 1, \qquad \forall e \in E(G).$$

(2) If G is an ℓ-diameter stable graph and x, y is a diametrical pair of vertices in G, then there are at least ℓ edge disjoint x, y-paths of length d(G) in G.

(3) If G is a graph with the property $g(e) = 3$, $\forall e \in E(G)$, then G is 2-diameter stable if and only if every pair of diametrical vertices in G is joined by at least two edge disjoint paths of length d(G).

(4) A graph G with d(G) = 2, 3 is 2-diameter stable if and only if there are at least two edge disjoint paths, of length not exceeding d(G), between any pair of vertices of G.

(5) For the functions $f_s(k,d)$, $g_s(k,d)$ and $g_s(n,k,d)$, defined in section 4.5, we have,

$$f_s(2,d) = \left[\frac{d+1}{2}\right] + d + 1, \qquad d \geq 2.$$

$$g_s(2,d) = 2(d+1).$$

$$1 \leq \lim_{n \to \infty} \frac{g_s(n,2,d)}{n} \leq \begin{cases} \dfrac{d}{d-2} , & \text{d-odd} \\ \\ \dfrac{d-1}{d-3} , & \text{d-even.} \end{cases}$$

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III. For ℓ-distance stable graphs we obtained an analog of the well known Menger Whitney Theorem (see [9], Chapter 5), as follows:

A graph G is ℓ-distance stable if and only if between any pair of nonadjacent vertices x, y ∈ V(G), there are at least ℓ edge disjoint paths of length $d_G(x,y)$.

From the variety of open problems directly arising from this research, we mention only a few:

(1) In (4.6) we had for an ℓ-diameter stable graph, G

girth(G) ≤ d(G) + 1 .

We could not decide whether (4.6) can be improved, or, whether for any arbitrary integers ℓ,d, one can always find an ℓ-diameter stable graph with diameter d and girth d + 1

(2) Finding a necessary and sufficient condition for a graph G, to be an (ℓ,d)-graph on an ℓ-diameter stable graph, remains an open problem. The sufficient condition in Theorem 4.14 for a graph to be a (2,d)-graph, does not seem to be a necessary condition for (2,d)-stability. But to prove that, a counter example is needed.

(3) Computation of the values of the functions $f_s(k,d)$, $g_s(k,d)$ and $g_s(n,k,d)$ for k > 2 (for k = 2, we obtained various results) is needed.

(4) Analogous problems for (ℓ,d)-stable, ℓ-diameter stable and ℓ-distance stable graph with respect to vertices may be obtained. For instance, a graph G is ℓ-distance stable (w.r. to vertices) if and only if between any pair of nonadjacent vertices of G there are at least ℓ-vertex disjoint x, y-paths of length $d_G(x,y)$ in G. We have not

pursued similar questions for (ℓ,d)-graphs and ℓ-diameter-stable graphs, with respect to vertices.

(5) Extremal problems analogous to those considered in this research may be posed for ℓ-distance stable graphs.

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